Chapter 1 : LOGARITHMS

Exercise 1.1

Example and Solutions

Ex. 1.1.1 : Solve : $\log_2 (7x + 2) = 3$

Soln. : Given equation contain logarithm on L.H.S.

$$\log_2 (7x + 2) = 3 \quad ...(1)$$

Use exponential formfor eqn. (1)...[$\log_a M = x \Rightarrow M = a^x$]

$\therefore 7x + 2 = (2)^3 = 8 \qquad \therefore 7x = 8 - 2 = 6$

$x = \frac{6}{7} \qquad \therefore \mathbf{x = \frac{6}{7}}$ ✓ ...Ans.

Ex. 1.1.2 (Q. 1(a), W-19, 2 Marks)
Find the value of x if $\log_3 (x + 6) = 2$.

Soln. : Given equation contain logarithm on L.H.S.

$\log_3 (x + 6) = 2$

Use exponential form for Equation (1)

...[$\log_a M = x \Rightarrow M = a^x$]

$\therefore \quad x + 6 = (3)^2 = 9$

$\therefore \quad x + 6 = 9$

$x = 9 - 6$

$x = 3$

$\therefore \quad x = 3$ ✓ ...Ans.

Ex.1.1.3 : Solve $\log_2 (3x + 7) = \log_2 (5x + 1)$

Soln. :

Given equation contains logarithm on both sides with same base,

$$\log_2 (3x + 7) = \log_2 (5x + 1) \quad ...(1)$$

Use Logarithm Equality Property for eqn. (1)

...[$\log_a M = \log_a N \Rightarrow M = N$]

$\therefore \; 3x + 7 = 5x + 1$

Collecting variables on L.H.S. and constants on R.H.S.

$\therefore \; 3x - 5x = 1 - 7 \qquad \therefore \; -2x = -6$

$x = \frac{-6}{-2} \qquad \therefore \mathbf{x = 3}$ ✓ ...Ans.

Ex. 1.1.4 : Use the rules of logarithms to simplify each of the following :

(i) $3 \log_3 2 - \log_3 4 + \log_3 \left(\frac{1}{2}\right)$

(ii) $5 \log_3 6 - (2 \log_3 4 + \log_3 18)$

Soln. :

(i) $3 \log_3 2 - \log_3 4 + \log_3 \left(\frac{1}{2}\right)$

By using logarithm of a power,

$3 \log_3 2 = \log_3 (2)^3 = \log_3 8$

Now, $3 \log_3 2 - \log_3 4 + \log_3 \left(\frac{1}{2}\right)$

$= \log_3 8 - \log_3 4 + \log_3 \left(\frac{1}{2}\right) \quad ...(1)$

Use Quotient rule of logarithm for eqn. (1)

...$\left[\log_a M - \log_a N = \log_a \left(\frac{M}{N}\right)\right]$

$= \log_3 \left(\frac{8}{4}\right) + \log_3 \left(\frac{1}{2}\right) = \log_3 (2) + \log_3 \left(\frac{1}{2}\right) \quad ...(2)$

Use Product rule of logarithm for eqn. (2)

...[$\log_a (MN) = \log_a M + \log_a N$]

$= \log_3 \left(2 \times \frac{1}{2}\right) = \log_3 1$

$\mathbf{3 \log_3 2 - \log_3 4 + \log_3 \left(\frac{1}{2}\right) = \log_3 1}$

$3 \log_3 2 - \log_3 4 + \log_3 \left(\frac{1}{2}\right) = 0$ ✓ ...[$\log_a 1 = 0$] ...Ans.

(ii) $5 \log_3 6 - (2 \log_3 4 + \log_3 18)$

Given : $5 \log_3 6 - (2 \log_3 4 + \log_3 18) \quad ...(1)$

Use Power rule of logarithm for eqn. (1)

...[$\log_a M^K = K \log_a M$]

$= \log_3 (6)^5 - [\log_3 (4)^2 + \log_3 18]$

$= \log_3 (6)^5 - [\log_3 (16) + \log_3 18] \quad ...(2)$

Use Product rule of logarithm for eqn. (2)

...[$\log_a M + \log_a N = \log_a (MN)$]

$= \log_3 (6)^5 - [\log_3 (16 \times 18)]$

$= \log_3 (7776) - \log_3 (288) \quad ...(3)$

Use Quotient rule of logarithm for eqn. (3)

...$\left[\log_a M - \log_a N = \log_a \left(\frac{M}{N}\right)\right]$

$= \log_3 \left(\frac{7776}{288}\right) = \log_3 (27)$

$5 \log_3 6 - (2 \log_3 4 + \log_3 18) = \log_3 27$

$= \log_3 (3)^3 \quad ...[27 = (3)^3]$

$\mathbf{= 3 \log_3 3} \quad ...[\log_a M^k = K \log_a M]$

$= 3\,(1) \quad ...[\log_a a = 1]$

$5 \log_3 6 - (2 \log_3 4 + \log_3 18) = \mathbf{3}$ ✓ ...Ans.

Ex. 1.1.5 : Solve : $\log_5 (4x + 11) = 2$

☑ **Soln. :**

Given, $\log_5(4x + 11) = 2$ ...(1)

▶ **Use for eqn. (1)** ...[**Exponential form : $a^n = N$ and its Logarithmic form : $n = \log_a N$**]

$4x + 11 = (5)^2$ $\therefore 4x + 11 = 25$

$4x = 25 - 11$ $\therefore 4x = 14$

$x = \frac{14}{4}$ $\therefore x = \frac{7}{2}$ ✓ ...**Ans.**

Ex. 1.1.6 : Solve : $\log_2 (x + 5) - \log_2 (2x - 1) = 5$

☑ **Soln. : Given,** $\log_2 (x + 5) - \log_2 (2x - 1) = 5$ ...(1)

▶ **Use Quotient rule of logarithm for eqn. (1)**

...[$\log_a M - \log_a N = \log_a \left(\frac{M}{N}\right)$]

Write logarithm in exponential form,

$\frac{x + 5}{2x - 1} = 2^5$ $\therefore \frac{x + 5}{2x - 1} = 32$

$x + 5 = 32 (2x - 1) = 64x - 32$

Collecting variables on L.H.S. and constants on R.H.S.

$x - 64x = -32 - 5$ $\therefore -63x = -37$

$x = \frac{-37}{-63}$ $\therefore$ $x = \frac{37}{63}$ ✓ ...**Ans.**

Ex. 1.1.7 : Solve : $\log_4 x + \log_4 (x - 12) = 3$

☑ **Soln. :**

Given, $\log_4 x + \log_4 (x - 12) = 3$ ...(1)

▶ **Use Product rule of logarithm for eqn. (1)**

...[$\log_a M + \log_a N = \log_a (MN)$]

$\log_4 (x (x - 12)) = 3$ ...(2)

▶ **Use for eqn. (2)** ...[**Exponential form : $a^n = N$ and its Logarithmic form : $n = \log_a N$**]

$x (x - 12) = (4)^3 = 64$

$x^2 - 12x = 64$ $\therefore x^2 - 12x - 64 = 0$

Which is quadratic equation,

$(x - 16)(x + 4) = 0$ [**Note the factors :** $(-16) + 4 = -12$ and $(-16) \times 4 = -64$]

$\therefore$ $x - 16 = 0$ OR $x + 4 = 0$

$\mathbf{x = 16}$ $\mathbf{x = -4}$

$x = -4$ is not possible,

Since logarithm of negative number is not defined

Hence solution is,

$\mathbf{x = 16}$ ✓ ...**Ans.**

Chapter Ends...

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Chapter 2 : DETERMINANT

Exercise 2.1

Ex. 2.1.1 (W-07, 2 Marks)

Find x, if $\begin{vmatrix} x & 2 \\ 8 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$

Soln. : Given, $\begin{vmatrix} x & 2 \\ 8 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$

Expand determinant of both sides.

$$\begin{vmatrix} x & 2 \\ 8 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$$

[Here order of determinant is 2]

[Product of elements in principal diagonal] – [Product of elements in secondary diagonal] = [Product of elements in principal diagonal] – [Product of elements in secondary diagonal]

$(x)(4) - (8)(2) = (1)(2) - (2)(1)$ (Refer the arrows)

$4x - 16 = 2 - 2$

$4x - 16 = 0$

$\therefore \quad 4x = 16; \quad \therefore \quad x = \frac{16}{4}$

$x = 4$ ✓ …Ans.

Ex. 2.1.2 (W-09, 2 Marks)

Solve, $\begin{vmatrix} x^2 & -x \\ -5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & -3 \\ 5 & -3 \end{vmatrix}$

Soln. :

Given : $\begin{vmatrix} x^2 & -x \\ -5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & -3 \\ 5 & -3 \end{vmatrix}$

Expand determinant of both sides,

$$\begin{vmatrix} x^2 & -x \\ -5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & -3 \\ 5 & -3 \end{vmatrix}$$

[Here order of determinant is 2]

[Product of elements in principal diagonal] – [Product of elements in secondary diagonal] = [Product of elements in principal diagonal] – [Product of elements in secondary diagonal]

$(x^2)(1) - (-5)(-x) = (7)(-3) - (5)(-3)$ (Refer arrows)

$x^2 - (5x) = (-21) - (-15)$

$x^2 - 5x = -21 + 15$

$x^2 - 5x = -6$

$x^2 - 5x + 6 = 0$

$(x-3)(x-2) = 0$ $\begin{bmatrix} \text{Note the factors :} \\ (-3) + (-2) = -5 \text{ and } (-3)(-2) = 6 \end{bmatrix}$

$\therefore \; x - 3 = 0 \quad \text{or} \quad x = 2$

x = 3 or x = 2 ✓ …Ans.

Ex. 2.1.3 (S-07, 2 Marks)

If A = $\begin{vmatrix} -1 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix}$ find |A|

Soln. : Given :

$$A = \begin{vmatrix} -1 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

Here order of determinant is 3

$$\therefore \; A = \begin{vmatrix} -1 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$A = (-1) \times [\text{Minor of } (-1)] - 1 \times [\text{Minor of } (1)] + (-1) \times [\text{Minor of } (-1)]$

$$= -1 \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} - (1) \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$$

$$A = (-1) \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} - 1 \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$$

$= (-1)[\underbrace{(3)(3)}_{9} - \underbrace{(2)(1)}_{2}] - 1[\underbrace{(2)(3)}_{6} - \underbrace{(1)(1)}_{1}] + (-1)[\underbrace{(2)(2)}_{4} - \underbrace{(1)(3)}_{3}]$

$= (-1)\underbrace{[9-2]}_{7} - 1\underbrace{[6-1]}_{5} + (-1)\underbrace{[4-3]}_{1}$

$= (-1)(7) - 1(5) - 1(1) = -7 - 5 - 1 = -13$ ✓ …Ans.

Ex. 2.1.4 (S-11, 2 Marks)

Find x if $\begin{vmatrix} 4 & 3 & 9 \\ 3 & 2 & 7 \\ 1 & 4 & x \end{vmatrix} = 0$

☑ **Soln. :**

Given : $\begin{vmatrix} 4 & 3 & 9 \\ 3 & 2 & 7 \\ 1 & 4 & x \end{vmatrix} = 0$

Here order of determinant is 3

$\therefore \begin{vmatrix} 4 & 3 & 9 \\ 3 & 2 & 7 \\ 1 & 4 & x \end{vmatrix} = 0$

$4 \times$ [Minor of 4] – $3 \times$ [Minor of 3] + $9 \times$ [Minor of 9] = 0

$4 \begin{vmatrix} 2 & 7 \\ 4 & x \end{vmatrix} - 3 \begin{vmatrix} 3 & 7 \\ 1 & x \end{vmatrix} + 9 \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 0$

$4[\underbrace{(2)(x)}_{2x} - \underbrace{(4)(7)}_{28}] - 3[\underbrace{(3)(x)}_{3x} - \underbrace{(1)(7)}_{7}]$

$+ 9[\underbrace{(3)(4)}_{12} - \underbrace{(1)(2)}_{2}] = 0$

$4[2x - 28] - 3[3x - 7] + 9\underbrace{(12 - 2)}_{10} = 0$ (by simplification)

$4(2x) - 4(28) - 3(3x) - 3(-7) + 9(10) = 0$

$8x - 112 - 9x + 21 + 90 = 0$

$\underbrace{(8x - 9x)}_{-x} + \underbrace{(-112 + 21 + 90)}_{-1} = 0$ [collecting terms containing x]

$-x + (-1) = 0$

$-x = 1$

$\mathbf{x = -1}$ ✓ …Ans.

Ex. 2.1.5 **(W-16, 2 Marks)**

Find x if $\begin{vmatrix} 1 & 1 & 1 \\ 3 & x & 3 \\ 1 & x & 2 \end{vmatrix} = 0$

☑ **Soln. :**

Given : $\begin{vmatrix} 1 & 1 & 1 \\ 3 & x & 3 \\ 1 & x & 2 \end{vmatrix} = 0$

Here order of determinant is 3

$\begin{vmatrix} 1 & 1 & 1 \\ 3 & x & 3 \\ 1 & x & 2 \end{vmatrix} = 0$

$1\times$ [Minor of 1] – $1\times$ [Minor of 1] + $1\times$ [Minor of 1] = **0**

$1 \begin{vmatrix} x & 3 \\ x & 2 \end{vmatrix} - 1 \begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & x \\ 1 & x \end{vmatrix} = 0$

$[\underbrace{(2)(x)}_{2x} - \underbrace{(3)(x)}_{3x}] - 1[\underbrace{(3)(2)}_{6} - \underbrace{(1)(3)}_{3}]$

$+ [\underbrace{(3)(x)}_{3x} - \underbrace{(1)(x)}_{x}] = 0$

$\underbrace{[2x - 3x]}_{-x} - \underbrace{[6 - 3]}_{3} + \underbrace{(3x - x)}_{2x} = 0$ (by simplification)

$-x - 3 + 2x = 0$

$\underbrace{(-x + 2x)}_{x} - 3 = 0$

$\mathbf{x = 3}$ ✓ …Ans.

Ex. 2.1.6 **(S-15, W-15, 2 Marks)**

Find x if $\begin{vmatrix} 4 & 3 & 9 \\ 3 & -2 & 7 \\ 11 & 4 & x \end{vmatrix} = 0$

☑ **Soln. :**

Given : $\begin{vmatrix} 4 & 3 & 9 \\ 3 & -2 & 7 \\ 11 & 4 & x \end{vmatrix} = 0$

Here order of determinant is 3

$\therefore \begin{vmatrix} 4 & 3 & 9 \\ 3 & -2 & 7 \\ 11 & 4 & x \end{vmatrix} = 0$

$4 \times$ [Minor of 4] – $3 \times$ [Minor of 3] + $9 \times$ [Minor of 9] = 0

$$4\begin{vmatrix}4&3&9\\3&-2&7\\11&4&x\end{vmatrix} -3\begin{vmatrix}4&3&9\\3&-2&7\\11&4&x\end{vmatrix} +9\begin{vmatrix}4&3&9\\3&-2&7\\11&4&x\end{vmatrix}=0$$

$$4\begin{vmatrix}-2&7\\4&x\end{vmatrix} -3\begin{vmatrix}3&7\\11&x\end{vmatrix} +9\begin{vmatrix}3&-2\\11&4\end{vmatrix}=0$$

$$4\,[\underbrace{(-2)(x)}_{-2x}-\underbrace{(4)(7)}_{28}]-3\,[\underbrace{(3)(x)}_{3x}-\underbrace{(11)(7)}_{77}]$$

$$+9\,[\underbrace{(3)(4)}_{12}-\underbrace{(11)(-2)}_{-22}]=0$$

$$4\,[-2x-28]-3\,[3x-77]+9\underbrace{(12+22)}_{34}=0 \quad \text{(by simplification)}$$

$$4(-2x)-4(28)-3(3x)-3(-77)+9(34)=0$$

$$-8x-112-9x+231+306=0$$

$$\underbrace{(-8x-9x)}_{-17x}+\underbrace{(-112+231+306)}_{425}=0 \quad \text{[collecting terms containing x]}$$

$$-17x+425 = 0$$

$$-17x = -425$$

$$x = \frac{-425}{-17}=25$$

$$x = 25 ✓$$ …Ans.

Ex. 2.1.7 (S-13, 2 Marks)

Find k if $\begin{vmatrix}2&-k&7\\3&-4&13\\8&-11&33\end{vmatrix}=0$

☑ **Soln. :**

Given : $\begin{vmatrix}2&-k&7\\3&-4&13\\8&-11&33\end{vmatrix}=0$

Here order of determinant is 3

$$\therefore \begin{vmatrix}2&-k&7\\3&-4&13\\8&-11&33\end{vmatrix}=0$$

$2\times$[Minor of 2] – (– k) × [Minor of (– k)] + 7 × [Minor of 7] = 0

$$2\begin{vmatrix}2&-k&7\\3&-4&13\\8&-11&33\end{vmatrix} -(-k)\begin{vmatrix}2&-k&7\\3&-4&13\\8&-11&33\end{vmatrix} +7\begin{vmatrix}2&-k&7\\3&-4&13\\8&-11&33\end{vmatrix}=0$$

$$\therefore 2\begin{vmatrix}-4&13\\-11&33\end{vmatrix} -(-k)\begin{vmatrix}3&13\\8&33\end{vmatrix} +7\begin{vmatrix}3&-4\\8&-11\end{vmatrix}=0$$

$$2\,[\underbrace{(-4)(33)}_{-132}-\underbrace{(-11)(13)}_{-143}]+k\,[\underbrace{(3)(33)}_{99}-\underbrace{(8)(13)}_{104}]$$

$$+7\,[\underbrace{(3)(-11)}_{-33}-\underbrace{(8)(-4)}_{-32}]=0$$

$$2\,[\underbrace{(-132)-(-143)}_{11}]+k\,[\underbrace{(99)-(104)}_{-5}]$$

$$+7\,[\underbrace{(-33)-(-32)}_{-1}]=0$$

$$2(11)+k(-5)+7(-1)=0 \quad \text{(by simplification)}$$

$$22-5k-7=0$$

$$\therefore -5k+(22-7) = 0$$

$$-5k+15 = 0$$

$$-5k = -15 \qquad \therefore k = \frac{-15}{-5}$$

$$k = 3 ✓$$ …Ans.

Ex. 2.1.8 (S-09, 2 Marks)

Solve $\begin{vmatrix}x&4&-4\\3&-2&1\\-2&-4&1\end{vmatrix}=0$

☑ **Soln. :**

Given : $\begin{vmatrix}x&4&-4\\3&-2&1\\-2&-4&1\end{vmatrix}=0$

Here order of determinant is 3.

$$\therefore \begin{vmatrix}x&4&-4\\3&-2&1\\-2&-4&1\end{vmatrix}=0$$

x×[Minor of x] –4×[Minor of 4] + (–4) ×[Minor of (–4)] = 0

$$x\begin{vmatrix}x&4&-4\\3&-2&1\\-2&-4&1\end{vmatrix} -(4)\begin{vmatrix}x&4&-4\\3&-2&1\\-2&-4&1\end{vmatrix} +(-4)\begin{vmatrix}x&4&-4\\3&-2&1\\-2&-4&1\end{vmatrix}=0$$

$$x\begin{vmatrix}-2&1\\-4&1\end{vmatrix} -4\begin{vmatrix}3&1\\-2&1\end{vmatrix} +(-4)\begin{vmatrix}3&-2\\-2&-4\end{vmatrix}=0$$

$$x\,[\underbrace{(-2)(1)}_{-2} - \underbrace{(-4)(1)}_{-4}] - 4\,[\underbrace{(3)(1)}_{3} - \underbrace{(-2)(1)}_{-2}]$$

$$-4\,[\underbrace{(3)(-4)}_{-12} - \underbrace{(-2)(-2)}_{4}] = 0$$

$$x\,[\underbrace{-2-(-4)}_{2}] - 4\,[\underbrace{3-(-2)}_{5}] - 4\,[\underbrace{-12-(4)}_{-16}] = 0$$

$$x(2) - 4(5) - 4(-16) = 0 \quad \text{(by simplification)}$$

$$2x - 20 + 64 = 0$$

$$2x + 44 = 0$$

$$2x = -44$$

$$x = \frac{-44}{2}$$

$$\mathbf{x = -22} ✓ \qquad \textbf{…Ans.}$$

Exercise 2.2

Ex. 2.2.1 **(S-10, 4 Marks)**

Solve by Cramer's rule : $3x + 3y - z = 11$, $2x - y + 2z = 9$ and $4x + 3y + 2z = 25$.

Soln. :

Step I : Given equations can be written as,

$$\left.\begin{aligned} 3(x) + 3(y) - 1(z) &= 11 \\ 2(x) - 1(y) + 2(z) &= 9 \\ 4(x) + 3(y) + 2(z) &= 25 \end{aligned}\right\} \qquad …(1)$$

By Cramer's Rule, $x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$ …(2)

Step II : Where,

$$D = \begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{vmatrix} \quad \begin{bmatrix}\text{Determinant of co-efficients} \\ \text{of x, y, z of Equation (1)}\end{bmatrix}$$

$$= \begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{vmatrix}$$

$= (3) \times [\text{Minor of } 3] - (3) \times [\text{Minor of } 3] + (-1) \times [\text{Minor of } (-1)]$

$$= 3 \begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} - 3 \begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}$$

$$= 3\,[\underbrace{(-1)(2)}_{-2} - \underbrace{(3)(2)}_{6}] - 3\,[\underbrace{(2)(2)}_{4} - \underbrace{(4)(2)}_{8}] - 1\,[\underbrace{(2)(3)}_{6} - \underbrace{(4)(-1)}_{-4}]$$

$$= 3\,[\underbrace{-2-6}_{-8}] - 3\,[\underbrace{4-8}_{-4}] - 1\,[\underbrace{6-(-4)}_{10}]$$

$$= 3(-8) - 3(-4) - 1(10) = -24 + 12 - 10$$

$$\therefore D = \mathbf{-22} \qquad …(3)$$

Step III :

$$D_x = \begin{vmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{vmatrix} \quad \begin{bmatrix}\text{In determinant D replace} \\ \text{co-efficientsof x by} \\ \text{constants of equation (1)}\end{bmatrix}$$

$$= \begin{vmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{vmatrix}$$

$= (11) \times [\text{Minor of } 11] - (3) \times [\text{Minor of } 3] + (-1) \times [\text{Minor of } (-1)]$

$$= 11 \begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} - 3 \begin{vmatrix} 9 & 2 \\ 25 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 9 & -1 \\ 25 & 3 \end{vmatrix}$$

$$= 11\,[\underbrace{(-1)(2)}_{-2} - \underbrace{(3)(2)}_{6}] - 3\,[\underbrace{(9)(2)}_{18} - \underbrace{(25)(2)}_{50}] - 1\,[\underbrace{(9)(3)}_{27} - \underbrace{(25)(-1)}_{-25}]$$

$$= 11\,[\underbrace{-2-6}_{-8}] - 3\,[\underbrace{18-50}_{-32}] - 1\,[\underbrace{27-(-25)}_{52}]$$

$$= 11(-8) - 3(-32) - 1(52) = -88 + 96 - 52$$

$$\therefore \mathbf{D_x = -44} \qquad …(4)$$

Step IV :

$$D_y = \begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix} \quad \begin{bmatrix}\text{In determinant D replace} \\ \text{co-efficients of y by} \\ \text{constants of Equations (1)}\end{bmatrix}$$

$$= \begin{vmatrix} \mathbf{3} & \mathbf{11} & \mathbf{-1} \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix}$$

$= (3) \times [\text{Minor of } 3] - (11) \times [\text{Minor of } 11] + (-1) \times [\text{Minor of } (-1)]$

$$= \mathbf{3}\begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix} - \mathbf{11}\begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix} + (-\mathbf{1})\begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix}$$

$$= \mathbf{3}\begin{vmatrix} 9 & 2 \\ 25 & 2 \end{vmatrix} - \mathbf{11}\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} + (-\mathbf{1})\begin{vmatrix} 2 & 9 \\ 4 & 25 \end{vmatrix}$$

$$= \mathbf{3}\,[\underbrace{(9)(2)}_{18} - \underbrace{(25)(2)}_{50}] - \mathbf{11}\,[\underbrace{(2)(2)}_{4} - \underbrace{(4)(2)}_{8}] - \mathbf{1}\,[\underbrace{(2)(25)}_{50} - \underbrace{(4)(9)}_{36}]$$

$$= \mathbf{3}\,[\underbrace{18 - 50}_{-32}] - \mathbf{11}\,[\underbrace{4 - 8}_{-4}] - \mathbf{1}\,[\underbrace{50 - 36}_{14}]$$

$= \mathbf{3}\,(-32) - \mathbf{11}\,(-4) - \mathbf{1}\,(14) = -96 + 44 - 14$

$\therefore \mathbf{D_y = -66}$ …(5)

Step V:

$$D_z = \begin{vmatrix} \mathbf{3} & \mathbf{3} & \mathbf{11} \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix}$$ [In determinant D replace co-efficients of z by constants of Equation (1)]

$$= \begin{vmatrix} \mathbf{3} & \mathbf{3} & \mathbf{11} \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix}$$

$= (3) \times [\text{Minor of } 3] - (3) \times [\text{Minor of } 1] + (11) \times [\text{Minor of } (11)]$

$$= \mathbf{3}\begin{vmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix} - \mathbf{3}\begin{vmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix} + \mathbf{11}\begin{vmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix}$$

$$= \mathbf{3}\begin{vmatrix} -1 & 9 \\ 3 & 25 \end{vmatrix} - \mathbf{3}\begin{vmatrix} 2 & 9 \\ 4 & 25 \end{vmatrix} + \mathbf{11}\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}$$

$$= \mathbf{3}\,[\underbrace{(-1)(25)}_{-25} - \underbrace{(3)(9)}_{27}] - \mathbf{3}\,[\underbrace{(2)(25)}_{50} - \underbrace{(4)(9)}_{36}] + \mathbf{11}\,[\underbrace{(2)(3)}_{6} - \underbrace{(4)(-1)}_{-4}]$$

$$= \mathbf{3}\,[\underbrace{-25 - 27}_{-52}] - \mathbf{3}\,[\underbrace{50 - 36}_{14}] + \mathbf{11}\,[\underbrace{6 - (-4)}_{10}]$$

$= \mathbf{3}\,(-52) - \mathbf{3}\,(14) + \mathbf{11}\,(10) = -156 - 42 + 110$

$\therefore \mathbf{D_z = -88}$ …(6)

Step VI : Hence,

$x = \dfrac{D_x}{D} = \dfrac{-44}{-22}$ [From Equations (2), (3), (4)]

$\mathbf{x = 2}$ ✓ **…Ans.**

$y = \dfrac{D_y}{D} = \dfrac{-66}{-22}$ [From Equations (2), (3), (5)]

$\mathbf{y = 3}$ ✓ **…Ans.**

$z = \dfrac{D_z}{D} = \dfrac{-88}{-22}$ [From Equations (2), (3), (6)]

$\mathbf{z = 4}$ ✓ **…Ans.**

Hence the solution is, **x = 2, y = 3, z = 4**

Ex. 2.2.2 (S-08, S-12, S-17, S-19, 4 Marks)

Solve the equation by Cramer's rule $x + y + z = 3$, $x - y + z = 1$, $x + y - 2z = 0$.

☑ **Soln. :**

Step I : Given equations can be written as,

$$\left.\begin{aligned} \mathbf{1}(x) + \mathbf{1}(y) + \mathbf{1}(z) &= 3 \\ \mathbf{1}(x) - \mathbf{1}(y) + \mathbf{1}(z) &= 1 \\ \mathbf{1}(x) + \mathbf{1}(y) - \mathbf{2}(z) &= 0 \end{aligned}\right\} \quad \ldots(1)$$

By Cramer's Rule, $x = \dfrac{D_x}{D}$, $y = \dfrac{D_y}{D}$, $z = \dfrac{D_z}{D}$ …(2)

Step II : Where, $D = \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix}$ [Determinant of co-efficients of x, y, z of Equation (1)]

$$= \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix}$$

$= (1) \times [\text{Minor of } 1] - (1) \times [\text{Minor of } 1] + (1) \times [\text{Minor of } (1)]$

$$= 1\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 1\,[\underbrace{(-1)(-2)}_{2} - \underbrace{(1)(1)}_{1}] - 1\,[\underbrace{(1)(-2)}_{-2} - \underbrace{(1)(1)}_{1}]$$

$$+ 1\,[\underbrace{(1)(1)}_{1} - \underbrace{(1)(-1)}_{-1}]$$

$$= 1[\underbrace{2-1}_{1}] - 1[\underbrace{-2-1}_{-3}] + 1[\underbrace{1-(-1)}_{2}]$$

$$= 1(1) - 1(-3) + 1(2) = 1 + 3 + 2$$

$$\therefore \mathbf{D} = \mathbf{6} \qquad \ldots(3)$$

▶ **Step III :** $D_x = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{vmatrix}$ [In determinant D replace, co-efficients of x by constants of Equation (1)]

$$= \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{vmatrix}$$

= (3) × [Minor of 3] – (1) × [Minor of 3] + (1) × [Minor of (1)]

$$= 3\begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= 3[\underbrace{(-1)(-2)}_{2} - \underbrace{(1)(1)}_{1}] - 1[\underbrace{(1)(-2)}_{-2} - \underbrace{(0)(1)}_{0}]$$

$$+ 1\,[\underbrace{(1)(1)}_{1} - \underbrace{(0)(-1)}_{0}]$$

$$= 3\,[\underbrace{2-1}_{1}] - 1[\underbrace{-2-0}_{-2}] + 1[\underbrace{1-0}_{1}]$$

$$= 3\,(1) - 1(-2) + 1(1) = 3 + 2 + 1$$

$$\therefore \mathbf{D_x} = \mathbf{6} \qquad \ldots(4)$$

▶ **Step IV :** $D_y = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{vmatrix}$ [In determinant D replace, co-efficients of y by constants of Equation (1)]

$$= \begin{vmatrix} 1 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{vmatrix}$$

= (1) × [Minor of 1] – (3) × [Minor of 3] + (1) × [Minor of (1)]

$$= 1\begin{vmatrix} 1 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 1[\underbrace{(1)(-2)}_{-2} - \underbrace{(0)(1)}_{0}] - 3\,[\underbrace{(1)(-2)}_{-2} - \underbrace{(1)(1)}_{1}]$$

$$+ 1[\underbrace{(1)(0)}_{0} - \underbrace{(1)(1)}_{1}]$$

$$= 1[\underbrace{-2-0}_{-2}] - 3[\underbrace{-2-1}_{-3}] + 1[\underbrace{0-1}_{-1}]$$

$$= 1(-2) - 3\,(-3) + 1(-1) = -2 + 9 - 1$$

$$\therefore \mathbf{D_y} = \mathbf{6} \qquad \ldots(5)$$

▶ **Step V :** $D_z = \begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$ [In determinant D replace, co-efficients of z by constants of Equation (1)]

$$= \begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

= (1) × [Minor of 1] – (1) × [Minor of 1] + (3) × [Minor of (3)]

$$= 1\begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 3 & 0 \\ 0 & 1 & -3 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 1\,[\underbrace{(-1)(0)}_{0} - \underbrace{(1)(1)}_{1}] - 1\,[\underbrace{(1)(0)}_{0} - \underbrace{(1)(1)}_{1}]$$

$$+ 3\,[\underbrace{(1)(1)}_{1} - \underbrace{(1)(-1)}_{-1}]$$

$$= 1\,[\underbrace{0-1}_{-1}] - 1\,[\underbrace{0-1}_{-1}] + 3\,[\underbrace{1-(-1)}_{2}]$$

$$= 1(-1) - 1(-1) + 3(2) = -1 + 1 + 6$$

$$\therefore D_z = 6 \qquad \ldots(6)$$

▸ **Step VI :** Hence,

$x = \dfrac{D_x}{D} = \dfrac{6}{6}$ [From Equations (2), (3), (4)]

x = 1✓ **…Ans.**

$y = \dfrac{D_y}{D} = \dfrac{6}{6}$ [From Equations (2), (3), (5)]

y = 1✓ **…Ans.**

$z = \dfrac{D_z}{D} = \dfrac{6}{6}$ [From Equations (2), (3), (6)]

z = 1✓ **…Ans.**

Hence the solution is, **x = 1, y = 1, z = 1**

Ex. 2.2.3 (W-09, 4 Marks)

Using Cramer's rule solve

$2x + 3y = 5,\ y - 3z = -2,\ z + 3x = 4.$

☑ **Soln. :**

▸ **Step I :** Given equations are,

$$2x + 3y = 5 \ ;\ y - 3z = -2 \ ;\ z + 3x = 4$$

Rewrite these equations as,

$$\left.\begin{aligned} 2(x) + 3(y) + 0(z) &= 5 \\ 0(x) + 1(y) - 3(z) &= -2 \\ 3(x) + 0(y) + 1(z) &= 4 \end{aligned}\right\} \qquad \ldots(1)$$

By Cramer's Rule, $x = \dfrac{D_x}{D},\ y = \dfrac{D_y}{D},\ z = \dfrac{D_z}{D}$ …(2)

▸ **Step II :** Where, $D = \begin{vmatrix} 2 & 3 & 0 \\ 0 & 1 & -3 \\ 3 & 0 & 1 \end{vmatrix}$ [Determinant of co-efficients of x,y,z of Equation (1)]

$$= \begin{vmatrix} 2 & 3 & 0 \\ 0 & 1 & -3 \\ 3 & 0 & 1 \end{vmatrix}$$

$= (2) \times [\text{Minor of } 2] - (3) \times [\text{Minor of } 3] + (0) \times [\text{Minor of } (0)]$

$$= 2\begin{vmatrix} 1 & -3 \\ 0 & 1 \end{vmatrix} - 3\begin{vmatrix} 0 & -3 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix}$$

$$= 2\,[\underbrace{(1)(1)}_{1} - \underbrace{(0)(-3)}_{0}] - 3\,[\underbrace{(0)(1)}_{0} - \underbrace{(3)(-3)}_{-9}] + 0$$

$$= 2\,[\underbrace{1-0}_{1}] - 3\,[\underbrace{0-(-9)}_{9}] = 2(1) - 3(9) = 2 - 27$$

$$\therefore D = -25 \qquad \ldots(3)$$

▸ **Step III :** $D_x = \begin{vmatrix} 5 & 3 & 0 \\ -2 & 1 & -3 \\ 4 & 0 & 1 \end{vmatrix}$ [In determinant D replace, co-efficients of x by constants of Equation (1)]

$$= \begin{vmatrix} 5 & 3 & 0 \\ -2 & 1 & -3 \\ 4 & 0 & 1 \end{vmatrix}$$

$= (5) \times [\text{Minor of } 5] - (3) \times [\text{Minor of } 3] + (0) \times [\text{Minor of } (0)]$

$$= 5\begin{vmatrix} 1 & -3 \\ 0 & 1 \end{vmatrix} - 3\begin{vmatrix} -2 & -3 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} -2 & 1 \\ 4 & 0 \end{vmatrix}$$

$$= 5\,[\underbrace{(1)(1)}_{1} - \underbrace{(0)(-3)}_{0}] - 3\,[\underbrace{(-2)(1)}_{-2} - \underbrace{(4)(-3)}_{-12}] + 0$$

$$= 5\,[\underbrace{1-0}_{1}] - 3\,[\underbrace{-2-(-12)}_{10}] = 5(1) - 3(10) = 5 - 30$$

$$\therefore D_x = -25 \qquad \ldots(4)$$

▶ **Step IV :** $D_y = \begin{vmatrix} \boldsymbol{2} & \boldsymbol{5} & \boldsymbol{0} \\ 0 & -2 & -3 \\ 3 & 4 & 1 \end{vmatrix}$ $\left[\begin{array}{l}\text{In determinant} \\ \text{D replace co-efficients} \\ \text{ofyby constants} \\ \text{of Equation (1)}\end{array}\right]$

$$= \begin{vmatrix} \boldsymbol{2} & \boldsymbol{5} & \boldsymbol{0} \\ 0 & -2 & -3 \\ 3 & 4 & 1 \end{vmatrix}$$

$= (2) \times [\text{Minor of } 2] - (5) \times [\text{Minor of } 5] + (0) \times [\text{Minor of } (0)]$

$$= \boldsymbol{2}\begin{vmatrix} -2 & -3 \\ 4 & 1 \end{vmatrix} \boldsymbol{-5}\begin{vmatrix} 0 & -3 \\ 3 & 1 \end{vmatrix} \boldsymbol{+0}\begin{vmatrix} 0 & -2 \\ 3 & 4 \end{vmatrix}$$

$$=\boldsymbol{2}\,[\underbrace{(-2)(1)}_{-2} - \underbrace{(4)(-3)}_{-12}] - \boldsymbol{5}[\underbrace{(0)(1)}_{0} - \underbrace{(3)(-3)}_{-9}] + \boldsymbol{0}$$

$$=\boldsymbol{2}[\underbrace{(-2)-(-12)}_{10}] - \boldsymbol{5}\,[\underbrace{0-(-9)}_{9}] = \boldsymbol{2}\,(10) - \boldsymbol{5}\,(9) = 20 - 45$$

$\therefore \mathbf{D_y = -25}$ …(5)

▶ **Step V :** $D_z = \begin{vmatrix} \boldsymbol{2} & \boldsymbol{3} & \boldsymbol{5} \\ 0 & 1 & -2 \\ 3 & 0 & 4 \end{vmatrix}$ $\left[\begin{array}{l}\text{In determinant} \\ \text{D replace,} \\ \text{co-efficients of} \\ \text{z by constants} \\ \text{of Equation (1)}\end{array}\right]$

$$= \begin{vmatrix} \boldsymbol{2} & \boldsymbol{3} & \boldsymbol{5} \\ 0 & 1 & -2 \\ 3 & 0 & 4 \end{vmatrix}$$

$= (2) \times [\text{Minor of } 2] - (3) \times [\text{Minor of } 3] + (5) \times [\text{Minor of } (5)]$

$$= \boldsymbol{2}\begin{vmatrix} 1 & -2 \\ 0 & 4 \end{vmatrix} \boldsymbol{-3}\begin{vmatrix} 0 & -2 \\ 3 & 4 \end{vmatrix} \boldsymbol{+5}\begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix}$$

$$=\boldsymbol{2}\,[\underbrace{(1)(4)}_{4} - \underbrace{(0)(-2)}_{0}] - \boldsymbol{3}\,[\underbrace{(0)(4)}_{0} - \underbrace{(3)(-2)}_{-6}] + \boldsymbol{5}\,[\underbrace{(0)(0)}_{0} - \underbrace{(3)(1)}_{3}]$$

$$=\boldsymbol{2}\,[\underbrace{4-0}_{4}] - \boldsymbol{3}\,[\underbrace{0-(-6)}_{6}] + \boldsymbol{5}\,[\underbrace{0-3}_{-3}]$$

$$=\boldsymbol{2}\,(4) - \boldsymbol{3}\,(6) + \boldsymbol{5}\,(-3) = 8 - 18 - 15$$

$\therefore \mathbf{D_z = -25}$ …(6)

▶ **Step VI :** Hence,

$x = \dfrac{D_x}{D} = \dfrac{-25}{-25}$ [From Equations (2), (3), (4)]

$\mathbf{x = 1}$ ✓ **…Ans.**

$y = \dfrac{D_y}{D} = \dfrac{-25}{-25}$ [From Equations (2), (3), (5)]

$\mathbf{y = 1}$ ✓ **…Ans.**

$z = \dfrac{D_z}{D} = \dfrac{-25}{-25}$ [From Equations (2), (3), (6)]

$\mathbf{z = 1}$ ✓ **…Ans.**

Hence the solution is, **x = 1, y = 1, z = 1**

Ex. 2.2.4 (W-11, W-12, 2 Marks)

Using Cramer's Rule solve : x + z = 4, y + z = 2, x + y = 0

☑ **Soln. :**

▶ **Step I :** Given equations are,

x + z = 4 ; y + z = 2 ; x + y = 0

These equations we can write as,

$$\left.\begin{array}{l} \boldsymbol{1}(x) + \boldsymbol{0}(y) + \boldsymbol{1}(z) = 4 \\ \boldsymbol{0}(x) + \boldsymbol{1}(y) + \boldsymbol{1}(z) = 2 \\ \boldsymbol{1}(x) + \boldsymbol{1}(y) + \boldsymbol{0}(z) = 0 \end{array}\right\} \quad \text{…(1)}$$

By Cramer's Rule, $x = \dfrac{D_x}{D}$, $y = \dfrac{D_y}{D}$; $z = \dfrac{D_z}{D}$ …(2)

▶ **Step II :** Where, $D = \begin{vmatrix} \boldsymbol{1} & \boldsymbol{0} & \boldsymbol{1} \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$ $\left[\begin{array}{c}\text{Determinant} \\ \text{of co-efficients} \\ \text{of x, y, z of} \\ \text{Equation (1)}\end{array}\right]$

$$= \begin{vmatrix} \boldsymbol{1} & \boldsymbol{0} & \boldsymbol{1} \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$= (1) \times [\text{Minor of } 1] - (0) \times [\text{Minor of } 0] + (1) \times [\text{Minor of } (1)]$

$$= 1\begin{vmatrix}1&0&1\\0&1&1\\1&1&0\end{vmatrix} - 0\begin{vmatrix}1&0&1\\0&1&1\\1&1&0\end{vmatrix} + 1\begin{vmatrix}1&0&1\\0&1&1\\1&1&0\end{vmatrix}$$

$$= 1\begin{vmatrix}1&1\\1&0\end{vmatrix} - 0\begin{vmatrix}1&0\\1&0\end{vmatrix} + 1\begin{vmatrix}0&1\\1&1\end{vmatrix}$$

$$= 1\,[\underbrace{(1)(0)}_{0} - \underbrace{(1)(1)}_{1}] - 0 + 1\,[\underbrace{(0)(1)}_{0} - \underbrace{(1)(1)}_{1}]$$

$$= 1\underbrace{[0-1]}_{-1} + 1\underbrace{[0-1]}_{-1} = 1\,(-1) + 1(-1) = -1-1$$

$\therefore \mathbf{D} = -2$ …(3)

Step III : $D_x = \begin{vmatrix}4&0&1\\2&1&1\\0&1&0\end{vmatrix}$ [In determinant D replace, co-efficients of x by constants of Equation (1)]

$$= \begin{vmatrix}4&0&1\\2&1&1\\0&1&0\end{vmatrix}$$

$= (4) \times [\text{Minor of } 4] - (0) \times [\text{Minor of } 0] + (1) \times [\text{Minor of } (1)]$

$$= 4\begin{vmatrix}4&0&1\\2&1&1\\0&1&0\end{vmatrix} - 0\begin{vmatrix}4&0&1\\2&1&1\\0&1&0\end{vmatrix} + 1\begin{vmatrix}4&0&1\\2&1&1\\0&1&0\end{vmatrix}$$

$$= 4\begin{vmatrix}1&1\\1&0\end{vmatrix} - 0\begin{vmatrix}2&1\\0&0\end{vmatrix} + 1\begin{vmatrix}2&1\\0&1\end{vmatrix}$$

$$= 4[\underbrace{(1)(0)}_{0} - \underbrace{(1)(1)}_{1}] - 0 + 1\,[\underbrace{(2)(1)}_{2} - \underbrace{(0)(1)}_{0}]$$

$$= 4\underbrace{[0-1]}_{-1} + 1\underbrace{[2-0]}_{2}\,4\,(-1) + 1\,(2) = -4+2$$

$\therefore \mathbf{D_x} = -2$ …(4)

Step IV : $D_y = \begin{vmatrix}1&4&1\\0&2&1\\1&0&0\end{vmatrix}$ [In determinant D replace, co-efficients of y by constants of Equation (1)]

$$= \begin{vmatrix}1&4&1\\0&2&1\\1&0&0\end{vmatrix}$$

$= (1) \times [\text{Minor of } 1] - (4) \times [\text{Minor of } 4] + (1) \times [\text{Minor of } (1)]$

$$= 1\begin{vmatrix}1&4&1\\0&2&1\\1&0&0\end{vmatrix} - 4\begin{vmatrix}1&4&1\\0&2&1\\1&0&0\end{vmatrix} + 1\begin{vmatrix}1&4&1\\0&2&1\\1&0&0\end{vmatrix}$$

$$= 1\begin{vmatrix}2&1\\0&0\end{vmatrix} - 4\begin{vmatrix}0&1\\1&0\end{vmatrix} + 1\begin{vmatrix}0&2\\1&0\end{vmatrix}$$

$$= 1[\underbrace{(2)(0)}_{0} - \underbrace{(0)(1)}_{0}] - 4\,[\underbrace{(0)(0)}_{0} - \underbrace{(1)(1)}_{1}]$$

$$+ 1\,[\underbrace{(0)(0)}_{0} - \underbrace{(1)(2)}_{2}]$$

$$= 1\underbrace{[0-0]}_{0} - 4\underbrace{[0-1]}_{-1} + 1\underbrace{[0-2]}_{-2} = 0 - 4\,(-1) + 1\,(-2)$$

$= 4 - 2$

$\therefore \mathbf{D_y} = 2$ …(5)

Step V : $D_z = \begin{vmatrix}1&0&4\\0&1&2\\1&1&0\end{vmatrix}$ [In determinant D replace, co-efficients of z by constants of Equation (1)]

$$= \begin{vmatrix}1&0&4\\0&1&2\\1&1&0\end{vmatrix}$$

$= (1) \times [\text{Minor of } 1] - (0) \times [\text{Minor of } 0] + (4) \times [\text{Minor of } (4)]$

$$= 1\begin{vmatrix}1&0&4\\0&1&2\\1&1&0\end{vmatrix} - 0\begin{vmatrix}1&0&4\\0&1&2\\1&1&0\end{vmatrix} + 4\begin{vmatrix}1&0&4\\0&1&2\\1&1&0\end{vmatrix}$$

$$= 1\begin{vmatrix}1&2\\1&0\end{vmatrix} - 0\begin{vmatrix}0&2\\1&0\end{vmatrix} + 4\begin{vmatrix}0&1\\1&1\end{vmatrix}$$

$$= 1\,[\underbrace{(1)(0)}_{0} - \underbrace{(1)(2)}_{2}] - 0 + 4\,[\underbrace{(0)(1)}_{0} - \underbrace{(1)(1)}_{1}]$$

$$= 1\,\underbrace{[0-2]}_{-2} + 4\,\underbrace{[0-1]}_{-1} = 1\,(-2) + 4\,(-1) = -2-4$$

$\therefore \mathbf{D_z} = -6$ …(6)

Step VI : Hence,

$$x = \frac{D_x}{D} = \frac{-2}{-2} \text{[From Equations (2), (3), (4)]}$$

$$\mathbf{x = 1} \checkmark \qquad \text{...Ans.}$$

$$y = \frac{D_y}{D} = \frac{2}{-2} \text{[From Equations (2), (3), (5)]}$$

$$\mathbf{y = -1} \checkmark \qquad \text{...Ans.}$$

$$z = \frac{D_z}{D} = \frac{-6}{-2} \text{[From Equations (2), (3), (6)]}$$

$$\mathbf{z = 3} \checkmark \qquad \text{...Ans.}$$

Hence the solution is, **x = 1, y = – 1, z = 3**

Ex. 2.2.5 (S-11, 4 Marks)

Using determinant method find x, if x – 2y + 3z = 4, 2x + y – 3z = 5, – x + y + 2z = 3.

Soln. :

Step I : Given equations can be written as,

$$\left.\begin{aligned} 1(x) - 2(y) + 3(z) &= 4 \\ 2(x) + 1(y) - 3(z) &= 5 \\ -1(x) + 1(y) + 2(z) &= 3 \end{aligned}\right\} \qquad \text{...(1)}$$

Step II : Where, $D = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -3 \\ -1 & 1 & 2 \end{vmatrix}$ [Determinant of co-efficients of x, y, z of Equation (1)]

$$= \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -3 \\ -1 & 1 & 2 \end{vmatrix}$$

= (1) × [Minor of 1] – (– 2) × [Minor of – 2] + (3) × [Minor of (3)]

$$= 1 \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -3 \\ -1 & 1 & 2 \end{vmatrix} -(-2) \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -3 \\ -1 & 1 & 2 \end{vmatrix} +3 \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -3 \\ -1 & 1 & 2 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & -3 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= 1\,[\underbrace{(1)(2)}_{2} - \underbrace{(1)(-3)}_{-3}] - 2\,[\underbrace{(2)(2)}_{4} - \underbrace{(-1)(-3)}_{3}] + 3\,[\underbrace{(2)(1)}_{2} - \underbrace{(-1)(1)}_{-1}]$$

$$= 1\,[\underbrace{2+3}_{5}] + 2\,[\underbrace{4-3}_{1}] + 3\,[\underbrace{2+1}_{3}] = 1\,(5) + 2\,(1) + 3\,(3)$$

$$= 5 + 2 + 9$$

$$\therefore \mathbf{D = 16} \qquad \text{...(3)}$$

Step III : $D_x = \begin{vmatrix} 4 & -2 & 3 \\ 5 & 1 & -3 \\ 3 & 1 & 2 \end{vmatrix}$ [In determinant D replace co-efficients of x by constants, of Equation (1)]

$$= \begin{vmatrix} 4 & -2 & 3 \\ 5 & 1 & -3 \\ 3 & 1 & 2 \end{vmatrix}$$

= (4) × [Minor of 4] – (– 2) × [Minor of – 2] + (3) × [Minor of (3)]

$$= 4 \begin{vmatrix} 4 & -2 & 3 \\ 5 & 1 & -3 \\ 3 & 1 & 2 \end{vmatrix} -(-2) \begin{vmatrix} 4 & -2 & 3 \\ 5 & 1 & -3 \\ 3 & 1 & 2 \end{vmatrix} +3 \begin{vmatrix} 4 & -2 & 3 \\ 5 & 1 & -3 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 1 & -3 \\ 1 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 5 & -3 \\ 3 & 2 \end{vmatrix} + 3 \begin{vmatrix} 5 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= 4\,[\underbrace{(1)(2)}_{2} - \underbrace{(1)(-3)}_{-3}] + 2\,[\underbrace{(5)(2)}_{10} - \underbrace{(3)(-3)}_{-9}] + 3\,[\underbrace{(5)(1)}_{5} - \underbrace{(3)(1)}_{3}]$$

$$= 4\,[\underbrace{2+3}_{5}] + 2\,[\underbrace{10+9}_{19}] + 3\,[\underbrace{5-3}_{2}]$$

$$= 4\,(5) + 2\,(19) + 3\,(2) = 20 + 38 + 6$$

$$\therefore \mathbf{D_x = 64} \qquad \text{...(4)}$$

Step IV : $D_y = \begin{vmatrix} 1 & 4 & 3 \\ 2 & 5 & -3 \\ -1 & 3 & 2 \end{vmatrix}$ [In determinant D replace co-efficients of y by constants, of Equation (1)]

$$= \begin{vmatrix} 1 & 4 & 3 \\ 2 & 5 & -3 \\ -1 & 3 & 2 \end{vmatrix}$$

= (1) × [Minor of 1] – (4) × [Minor of 4] + (3) × [Minor of (3)]

$$= 1 \begin{vmatrix} 1 & 4 & 3 \\ 2 & 5 & -3 \\ -1 & 3 & 2 \end{vmatrix} - 4 \begin{vmatrix} 1 & 4 & 3 \\ 2 & 5 & -3 \\ -1 & 3 & 2 \end{vmatrix} + 3 \begin{vmatrix} 1 & 4 & 3 \\ 2 & 5 & -3 \\ -1 & 3 & 2 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 5 & -3 \\ 3 & 2 \end{vmatrix} - 4 \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} + 3 \begin{vmatrix} 2 & 5 \\ -1 & 3 \end{vmatrix}$$

$$= 1\,[\underbrace{(5)(2)}_{10} - \underbrace{(3)(-3)}_{-9}] - 4\,[\underbrace{(2)(2)}_{4} - \underbrace{(-1)(-3)}_{3}] + 3\,[\underbrace{(2)(3)}_{6} - \underbrace{(-1)(5)}_{-5}]$$

$= \boldsymbol{1}\,[\underbrace{10+9}_{19}] - \boldsymbol{4}\,[\underbrace{4-3}_{1}] + \boldsymbol{3}\,[\underbrace{6+5}_{11}] = \boldsymbol{1}\,(19) - \boldsymbol{4}\,(1) + \boldsymbol{3}\,(11)$

$= 19 - 4 + 33$

$\therefore \mathbf{D_y = 48}$ …(5)

▶ **Step V :** $D_z = \begin{vmatrix} \boldsymbol{1} & \boldsymbol{-2} & \boldsymbol{4} \\ 2 & 1 & 5 \\ -1 & 1 & 3 \end{vmatrix}$ [In determinant D replace co-efficients of z by constants, of Equation (1)]

$$= \begin{vmatrix} \boldsymbol{1} & \boldsymbol{-2} & \boldsymbol{4} \\ 2 & 1 & 5 \\ -1 & 1 & 3 \end{vmatrix}$$

$= (1) \times [\text{Minor of } 1] - (-2) \times [\text{Minor of } -2] + (4) \times [\text{Minor of } (4)]$

$$= \boldsymbol{1}\begin{vmatrix} 1 & -2 & 4 \\ 2 & 1 & 5 \\ -1 & 1 & 3 \end{vmatrix} \boldsymbol{-(-2)}\begin{vmatrix} 1 & -2 & 4 \\ 2 & 1 & 5 \\ -1 & 1 & 3 \end{vmatrix} \boldsymbol{+4}\begin{vmatrix} 1 & -2 & 4 \\ 2 & 1 & 5 \\ -1 & 1 & 3 \end{vmatrix}$$

$$= \boldsymbol{1}\begin{vmatrix} 1 & 5 \\ 1 & 3 \end{vmatrix} - (-\boldsymbol{2})\begin{vmatrix} 2 & 5 \\ -1 & 3 \end{vmatrix} + \boldsymbol{4}\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}$$

$= \boldsymbol{1}\,[\underbrace{(1)\,(3)}_{3} - \underbrace{(1)\,(5)}_{5}] + \boldsymbol{2}\,[\underbrace{(2)\,(3)}_{6} - \underbrace{(-1)\,(5)}_{-5}]$

$+ \boldsymbol{4}\,[\underbrace{(2)\,(1)}_{2} - \underbrace{(-1)\,(1)}_{-1}] = \boldsymbol{1}\,[\underbrace{3-5}_{-2}] + \boldsymbol{2}\,[\underbrace{6+5}_{11}] + \boldsymbol{4}\,[\underbrace{2+1}_{3}]$

$= \boldsymbol{1}\,(-2) + \boldsymbol{2}\,(11) + \boldsymbol{4}\,(3) = -2 + 22 + 12$

$\therefore \mathbf{D_z = 32}$ …(6)

▶ **Step VI :** Hence,

$x = \dfrac{D_x}{D} = \dfrac{64}{16}$ [From Equations (2), (3), (4)]

$\mathbf{x = 4}$ ✓ **…Ans.**

$y = \dfrac{D_y}{D} = \dfrac{48}{16}$ [From Equations (2), (3), (5)]

$\mathbf{y = 3}$ ✓ **…Ans.**

$z = \dfrac{D_z}{D} = \dfrac{32}{16}$ [From Equations (2), (3), (6)]

$\mathbf{z = 2}$ ✓ **…Ans.**

Hence the solution is, **x = 4, y = 3, z = 2**

Note : To find x evaluate D and D_x only.

Ex. 2.2.6 **S-07, 4 Marks)**

Using Cramer's Rule solve the equations

$2x + 4z = 5y + 28$; $x + 11y = 5z - 41$, $3x - 3 = 2y + z$

☑ **Soln. :**

▶ **Step I :** Given equations are,

$2x + 4z = 5y + 28$; $x + 11y = 5z - 41$; $3x - 3 = 2y + z$

Rewrite equations as,

$$\left.\begin{aligned} \boldsymbol{2}(x) - \boldsymbol{5}(y) + \boldsymbol{4}(z) &= 28 \\ \boldsymbol{1}(x) + \boldsymbol{11}(y) - \boldsymbol{5}(z) &= -41 \\ \boldsymbol{3}(x) - \boldsymbol{2}(y) - \boldsymbol{1}(z) &= 3 \end{aligned}\right\} \quad \text{…(1)}$$

By Cramer's Rule, $x = \dfrac{D_x}{D}$; $y = \dfrac{D_y}{D}$; $z = \dfrac{D_z}{D}$ …(2)

▶ **Step II :** where, $D = \begin{vmatrix} \boldsymbol{2} & \boldsymbol{-5} & \boldsymbol{4} \\ 1 & 11 & -5 \\ 3 & -2 & -1 \end{vmatrix}$ [Determinant of co-efficients of x, y, z of Equation (1)]

$$= \begin{vmatrix} \boldsymbol{2} & \boldsymbol{-5} & \boldsymbol{4} \\ 1 & 11 & -5 \\ 3 & -2 & -1 \end{vmatrix}$$

$= (2) \times [\text{Minor of } 2] - (-5) \times [\text{Minor of } -5] + (4) \times [\text{Minor of } (4)]$

$$= \boldsymbol{2}\begin{vmatrix} 2 & -5 & 4 \\ 1 & 11 & -5 \\ 3 & -2 & -1 \end{vmatrix} \boldsymbol{-(-5)}\begin{vmatrix} 2 & -5 & 4 \\ 1 & 11 & -5 \\ 3 & -2 & -1 \end{vmatrix} \boldsymbol{+4}\begin{vmatrix} 2 & -5 & 4 \\ 1 & 11 & -5 \\ 3 & -2 & -1 \end{vmatrix}$$

$$= \boldsymbol{2}\begin{vmatrix} 11 & -5 \\ -2 & -1 \end{vmatrix} + \boldsymbol{5}\begin{vmatrix} 1 & -5 \\ 3 & -1 \end{vmatrix} + \boldsymbol{4}\begin{vmatrix} 1 & 11 \\ 3 & -2 \end{vmatrix}$$

$= \boldsymbol{2}\,[\underbrace{(11)\,(-1)}_{-11} - \underbrace{(-2)\,(-5)}_{10}] + \boldsymbol{5}\,[\underbrace{(1)\,(-1)}_{-1} - \underbrace{(3)\,(-5)}_{-15}]$

$+ \boldsymbol{4}\,[\underbrace{(1)\,(-2)}_{-2} - \underbrace{(3)\,(11)}_{33}]$

$= \boldsymbol{2}\,[\underbrace{-11-10}_{-21}] + \boldsymbol{5}[\underbrace{-1+15}_{14}] + \boldsymbol{4}[\underbrace{-2-33}_{-35}]$

$= \boldsymbol{2}\,(-21) + \boldsymbol{5}\,(14) + \boldsymbol{4}\,(-35) = -42 + 70 - 140$

$\therefore \mathbf{D = -112}$ …(3)

▶ **Step III :** $D_x = \begin{vmatrix} \boldsymbol{28} & \boldsymbol{-5} & \boldsymbol{4} \\ -41 & 11 & -5 \\ 3 & -2 & -1 \end{vmatrix}$ [In determinant D replace co-efficients of x by constants, of Equation (1)]

$$= \begin{vmatrix} 28 & -5 & 4 \\ -41 & 11 & -5 \\ 3 & -2 & -1 \end{vmatrix}$$

$= (28) \times [\text{Minor of } 28] - (-5) \times [\text{Minor of } -5]$
$+ (4) \times [\text{Minor of } (4)]$

$$= 28 \begin{vmatrix} 28 & -5 & 4 \\ -41 & 11 & -5 \\ 3 & -2 & -1 \end{vmatrix} - (-5) \begin{vmatrix} 28 & -5 & 4 \\ -41 & 11 & -5 \\ 3 & -2 & -1 \end{vmatrix} + 4 \begin{vmatrix} 28 & -5 & 4 \\ -41 & 11 & -5 \\ 3 & -2 & -1 \end{vmatrix}$$

$$= 28 \begin{vmatrix} 11 & -5 \\ -2 & -1 \end{vmatrix} - 5 \begin{vmatrix} -41 & -5 \\ 3 & -1 \end{vmatrix} + 4 \begin{vmatrix} -41 & 11 \\ 3 & -2 \end{vmatrix}$$

$$= 28[\underbrace{(11(-1)}_{-11} - \underbrace{(-2)(-5)}_{10}] + 5[\underbrace{(-41)(-1)}_{41} - \underbrace{(3)(-5)}_{-15}]$$

$$+ 4[\underbrace{(-41)(-2)}_{82} - \underbrace{(3)(11)}_{33}]$$

$$= 28[\underbrace{-11 - 10}_{-21}] + 5[\underbrace{41 + 15}_{56}] + 4[\underbrace{82 - 33}_{49}]$$

$= 28(-21) + 5(56) + 4(49)$

$= -588 + 280 + 196$

$\therefore \mathbf{D_x = -112}$ …(4)

▸ Step IV : $D_y = \begin{vmatrix} 2 & 28 & 4 \\ 1 & -41 & -5 \\ 3 & 3 & -1 \end{vmatrix}$ [In determinant D replace co-efficients of y by constants, of Equation (1)]

$$= \begin{vmatrix} 2 & 28 & 4 \\ 1 & -41 & -5 \\ 3 & 3 & -1 \end{vmatrix}$$

$= (2) \times [\text{Minor of } 2] - (28) \times [\text{Minor of } 28]$
$+ (4) \times [\text{Minor of } (4)]$

$$= 2 \begin{vmatrix} 2 & 28 & 4 \\ 1 & -41 & -5 \\ 3 & 3 & -1 \end{vmatrix} - 28 \begin{vmatrix} 2 & 28 & 4 \\ 1 & -41 & -5 \\ 3 & 3 & -1 \end{vmatrix} + 4 \begin{vmatrix} 2 & 28 & 4 \\ 1 & -41 & -5 \\ 3 & 3 & -1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} -41 & -5 \\ 3 & -1 \end{vmatrix} - 28 \begin{vmatrix} 1 & -5 \\ 3 & -1 \end{vmatrix} + 4 \begin{vmatrix} 1 & -41 \\ 3 & 3 \end{vmatrix}$$

$$= 2[\underbrace{(-41)(-1)}_{41} - \underbrace{(3)(-5)}_{-15}] - 28[\underbrace{(1)(-1)}_{-1} - \underbrace{(3)(-5)}_{-15}]$$

$$+ 4[\underbrace{(1)(3)}_{3} - \underbrace{(3)(-41)}_{-123}]$$

$$= 2[\underbrace{41 + 15}_{56}] - 28[\underbrace{-1 + 15}_{14}] + 4[\underbrace{3 + 123}_{126}]$$

$= 2(56) - 28(14) + 4(126) = 112 - 392 + 504$

$\therefore \mathbf{D_y = 224}$ …(5)

▸ Step V : $D_z = \begin{vmatrix} 2 & -5 & 28 \\ 1 & 11 & -41 \\ 3 & -2 & 3 \end{vmatrix}$ [In determinant D replace co-efficients of z by constants, of Equation (1)]

$$= \begin{vmatrix} 2 & -5 & 28 \\ 1 & 11 & -41 \\ 3 & -2 & 3 \end{vmatrix}$$

$= (2) \times [\text{Minor of } 2] - (-5) \times [\text{Minor of } -5]$
$+ (28) \times [\text{Minor of } 28]$

$$= 2 \begin{vmatrix} 2 & -5 & 28 \\ 1 & 11 & -41 \\ 3 & -2 & 3 \end{vmatrix} - (-5) \begin{vmatrix} 2 & -5 & 28 \\ 1 & 11 & -41 \\ 3 & -2 & 3 \end{vmatrix} + 28 \begin{vmatrix} 2 & -5 & 28 \\ 1 & 11 & -41 \\ 3 & -2 & 3 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 11 & -41 \\ -2 & 3 \end{vmatrix} - 5 \begin{vmatrix} 1 & -41 \\ 3 & 3 \end{vmatrix} + 28 \begin{vmatrix} 1 & 11 \\ 3 & -2 \end{vmatrix}$$

$$= 2[\underbrace{(11)(3)}_{33} - \underbrace{(-2)(-41)}_{82}] + 5[\underbrace{(1)(3)}_{3} - \underbrace{(3)(-41)}_{-123}]$$

$$+ 28[\underbrace{(1)(-2)}_{-2} - \underbrace{(3)(11)}_{33}]$$

$$= 2[\underbrace{33 - 82}_{-49}] + 5[\underbrace{3 + 123}_{126}] + 28[\underbrace{-2 - 33}_{-35}]$$

$= 2(-49) + 5(126) + 28(-35) = -98 + 630 - 980$

$\therefore \mathbf{D_z = -448}$ …(6)

▸ Step VI : Hence,

$x = \dfrac{D_x}{D} = \dfrac{-112}{-112}$ [From Equations (2) (3), (4)]

$\mathbf{x = 1}$ ✓ **…Ans.**

$$y = \frac{D_y}{D} = \frac{224}{-112} \quad \text{[From Equations (2), (3), (5)]}$$

y = – 2 ✓ …Ans.

$$z = \frac{D_z}{D} = \frac{-448}{-112} \quad \text{[From Equations (2), (3), (6)]}$$

z = 4 ✓ …Ans.

Hence the solution is, **x = 1, y = –2, z = 4**

Ex. 2.2.7 W-08, 4 Marks)

Find the area of the triangle ABC whose vertices are A (1, 1), B (2, 1), C (– 3, 2)

☑ **Soln. :**

We know, area of triangle ABC with vertices A (x_1, y_1), B (x_2, y_2), C (x_3, y_3) is,

$$\therefore \text{ Area of } \Delta \text{ ABC} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Given, A $(x_1, y_1) \equiv (1, 1)$, B $(x_2, y_2) \equiv (2, 1)$, C $(x_3, y_3) \equiv (-3, 2)$.

$$\therefore \text{ Area of } \Delta \text{ ABC} = \pm\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -3 & 2 & 1 \end{vmatrix}$$

$$= \pm\frac{1}{2}\left\{ 1\times\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 1\times\begin{vmatrix} 2 & 1 \\ -3 & 1 \end{vmatrix} + 1\times\begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix}\right\}$$

$$= \pm\frac{1}{2}\{1[\underbrace{(1)(1)}_{1} - \underbrace{(2)(1)}_{2}] - 1[\underbrace{(2)(1)}_{2} - \underbrace{(-3)(1)}_{-3}]$$

$$+ 1[\underbrace{(2)(2)}_{4} - \underbrace{(-3)(1)}_{-3}]\}$$

$$= \pm\frac{1}{2}\{\underbrace{(1-2)}_{-1} - \underbrace{(2+3)}_{5} + \underbrace{(4+3)}_{7}\} = \pm\frac{1}{2}\{-1-5+7\}$$

$$= \pm\frac{1}{2}(1) = \frac{1}{2}(1) \quad \text{(Since area is positive)}$$

Area of Δ ABC = $\frac{1}{2}$ square units. ✓ …Ans.

Ex. 2.2.8 (Q. 1(b), W-18, 2 Marks)

Find the area of the triangle whose vertices are (4, 3) (1, 4) and (2, 3).

☑ **Soln. :**

We know, area of triangle ABC with vertices A (x_1, y_1), B (x_2, y_2), C (x_3, y_3) is,

$$\therefore \text{ Area of } \Delta \text{ ABC} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Given, A $(x_1, y_1) \equiv (4, 3)$

B $(x_2, y_2) \equiv (1, 4)$

C $(x_3, y_3) \equiv (2, 3)$

$$\therefore \text{ Area of } \Delta \text{ ABC} = \pm\frac{1}{2}\begin{vmatrix} 4 & 3 & 1 \\ 1 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= \pm\frac{1}{2}\left\{ 4\times\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} - 3\times\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 1\times\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}\right\}$$

$$= \pm\frac{1}{2}\{4[\underbrace{(4)(1)}_{4} - \underbrace{(3)(1)}_{3}] - 3[\underbrace{(1)(1)}_{1} - \underbrace{(2)(1)}_{2}]$$

$$+ 1[\underbrace{(1)(3)}_{3} - \underbrace{(2)(4)}_{8}]\}$$

$$= \pm\frac{1}{2}\{\underbrace{(4-3)}_{1} - 3\underbrace{(1-2)}_{-1} + \underbrace{(3+8)}_{11}\} = \pm\frac{1}{2}\{1+3+11\}$$

$$= \pm\frac{1}{2}(15) = \frac{1}{2}(15) \quad \text{(Since area is positive)}$$

∴ Area of Δ ABC = $\frac{15}{2}$ square units. ✓ …Ans.

Chapter Ends...

❑❑❑

Chapter 3 : MATRICES

EXERCISE 3.1

Ex. 3.1.1 W-10, 2 Marks.

If $X = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ and $Y = \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$ find 3X + Y.

☑ **Soln. :** Given matrices,

Matrix	Order	
$X = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$	2×2	…(1)
$Y = \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$	2×2	…(2)

Order of X and Y is same.

∴ 3X + Y exist.

Now, $3X + Y = 3\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$ [From Equations (1) and (2)]

$= \begin{bmatrix} 3\times 1 & 3\times 2 \\ 3\times(-3) & 3\times 4 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$ [By scalar multiplication]

$= \begin{bmatrix} 3 & 6 \\ -9 & 12 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$

$= \begin{bmatrix} 3+4 & 6+5 \\ -9+1 & 12+(-3) \end{bmatrix}$ [Aaddition of two matrices]

$\mathbf{3X + Y} = \begin{bmatrix} \mathbf{7} & \mathbf{11} \\ \mathbf{-8} & \mathbf{9} \end{bmatrix}$ ✓ This is required matrix.

Ex. 3.1.2 S-11, 2 Marks.

If $A = \begin{bmatrix} 2 & 3 & 2 \\ 0 & -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \end{bmatrix}$

evaluate 3A – 4B.

☑ **Soln. :**

Given matrices are,

Matrix	Order	
$A = \begin{bmatrix} 2 & 3 & 2 \\ 0 & -1 & 5 \end{bmatrix}$	2×3	…(1)
$B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \end{bmatrix}$	2×3	…(2)

Here order of matrix A and matrix B is same

∴3A – 4B exist

Now, $3A - 4B = 3\begin{bmatrix} 2 & 3 & 2 \\ 0 & -1 & 5 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \end{bmatrix}$ [by scalar multiplication]

$= \begin{bmatrix} 3\times 2 & 3\times 3 & 3\times 2 \\ 3\times 0 & 3\times(-1) & 3\times 5 \end{bmatrix} - \begin{bmatrix} 4\times 1 & 4\times 2 & 4\times 1 \\ 4\times 0 & 4\times(-1) & 4\times 3 \end{bmatrix}$

$= \begin{bmatrix} 6 & 9 & 6 \\ 0 & -3 & 15 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 4 \\ 0 & -4 & 12 \end{bmatrix}$

$= \begin{bmatrix} 6-4 & 9-8 & 6-4 \\ 0-0 & -3-(-4) & 15-12 \end{bmatrix}$ (subtraction of two matrices)

$\mathbf{3A - 4B} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{3} \end{bmatrix}$ ✓ This is required matrix

Ex. 3.1.3 W - 11, 4 Marks, W -12, 2 Marks.

If $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$

(i) Find 2A + 3B – 4I, Where I is the unit matrix

(ii) Find 3A – 2B

☑ **Soln. :** Given matrices,

Matrix	Order	
$A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$	2×2	…(1)
$B = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$	2×2	…(2)

Also unit matrix I is,

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with same order as 2×2 …(3)

(i) Since orders of matrices A, B and I are same.

∴ 2A + 3B – 4I exist.

$\therefore 2A + 3B - 4I = 2\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} + 3\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

[from Equations (1), (2), (3)]

$= \begin{bmatrix} 2\times 2 & 2\times 3 \\ 2\times 4 & 2\times 7 \end{bmatrix} + \begin{bmatrix} 3\times 1 & 3\times 3 \\ 3\times 4 & 3\times 6 \end{bmatrix} - \begin{bmatrix} 4\times 1 & 4\times 0 \\ 4\times 0 & 4\times 1 \end{bmatrix}$

(by scalar multiplication)

$= \begin{bmatrix} 4 & 6 \\ 8 & 14 \end{bmatrix} + \begin{bmatrix} 3 & 9 \\ 12 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 4+3 & 6+9 \\ 8+12 & 14+18 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad \begin{bmatrix}\text{Addition of first} \\ \text{two matrices}\end{bmatrix}$$

$$= \begin{bmatrix} 7 & 15 \\ 20 & 32 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7-4 & 15-0 \\ 20-0 & 32-4 \end{bmatrix} \quad \text{[subtraction of two matrices]}$$

$$\mathbf{2A + 3B - 4I} = \begin{bmatrix} \mathbf{3} & \mathbf{15} \\ \mathbf{20} & \mathbf{28} \end{bmatrix} ✓$$ This is required matrix.

(ii) $3A - 2B = 3\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} - 2\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$ $\begin{bmatrix}\text{From Equations} \\ \text{(1) and (2)}\end{bmatrix}$

$$= \begin{bmatrix} 3\times 2 & 3\times 3 \\ 3\times 4 & 3\times 7 \end{bmatrix} - \begin{bmatrix} 2\times 1 & 2\times 3 \\ 2\times 4 & 2\times 6 \end{bmatrix} \quad \left(\begin{matrix}\text{by scalar} \\ \text{multiplication}\end{matrix}\right)$$

$$= \begin{bmatrix} 6 & 9 \\ 12 & 21 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 8 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 6-2 & 9-6 \\ 12-8 & 21-12 \end{bmatrix} \quad \text{(subtraction of two matrices)}$$

$$\mathbf{3A - 2B} = \begin{bmatrix} \mathbf{4} & \mathbf{3} \\ \mathbf{4} & \mathbf{9} \end{bmatrix} ✓$$ This is required matrix.

Ex. 3.1.4 (W-16, 2 Marks)

If $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ find $2A + 3B - 5I$, where I is unit matrix of order two :

☑ **Soln. :** Given equations are,

Matrix	Order	
$A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$	2×2	…(1)
$B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$	2×2	…(2)

Here order of matrices A, B are same

Also, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ unit matrix of order 2 …(3)

Now,

$$2A + 3B - 5I = 2\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} + 3\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

[From Equations (1), (2) and (3)]

$$= \begin{bmatrix} 2\times 2 & 2\times 3 \\ 2\times 4 & 2\times 7 \end{bmatrix} + \begin{bmatrix} 3\times 1 & 3\times 3 \\ 3\times(-2) & 3\times 5 \end{bmatrix} - \begin{bmatrix} 5\times 1 & 5\times 0 \\ 5\times 0 & 5\times 1 \end{bmatrix}$$ …[By scalar multiplication]

$$= \begin{bmatrix} 4 & 6 \\ 8 & 14 \end{bmatrix} + \begin{bmatrix} 3 & 9 \\ -6 & 15 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3 & 6+9 \\ 8+(-6) & 14+15 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

(Addition of 1^{st} two matrices)

$$= \begin{bmatrix} 7 & 15 \\ 2 & 29 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 7-5 & 15-0 \\ 2-0 & 29-5 \end{bmatrix} = \begin{bmatrix} 2 & 15 \\ 2 & 24 \end{bmatrix}$$

$$\mathbf{2A = 3B - 5I} = \begin{bmatrix} \mathbf{2} & \mathbf{15} \\ \mathbf{2} & \mathbf{24} \end{bmatrix} ✓$$ **…Ans.**

Ex. 3.1.5 S-10, 4 Marks.

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 7 & 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 3 \\ 4 & 0 & -1 \\ 2 & 3 & 1 \end{bmatrix}$ find $2A - 3B$.

☑ **Soln. :**

Given matrices are,

Matrix	Order	
$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 7 & 8 & 9 \end{bmatrix}$	3×3	…(1)
$B = \begin{bmatrix} 2 & 0 & 3 \\ 4 & 0 & -1 \\ 2 & 3 & 1 \end{bmatrix}$	3×3	…(2)

Since orders of matrix A and matrix B are same.

∴ 2A – 3B exist,

$$2A - 3B = 2\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 7 & 8 & 9 \end{bmatrix} - 3\begin{bmatrix} 2 & 0 & 3 \\ 4 & 0 & -1 \\ 2 & 3 & 1 \end{bmatrix}$$

(from Equations (1) and (2))

$$= \begin{bmatrix} 2\times 1 & 2\times 2 & 2\times 3 \\ 2\times 0 & 2\times 4 & 2\times 5 \\ 2\times 7 & 2\times 8 & 2\times 9 \end{bmatrix} - \begin{bmatrix} 3\times 2 & 3\times 0 & 3\times 3 \\ 3\times 4 & 3\times 0 & 3\times(-1) \\ 3\times 2 & 3\times 3 & 3\times 1 \end{bmatrix}$$

(by scalar multiplication)

$$= \begin{bmatrix} 2 & 4 & 6 \\ 0 & 8 & 10 \\ 14 & 16 & 18 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 9 \\ 12 & 0 & -3 \\ 6 & 9 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2-6 & 4-0 & 6-9 \\ 0-12 & 8-0 & 10-(-3) \\ 14-6 & 16-9 & 18-3 \end{bmatrix} \quad \left(\begin{matrix}\text{subtraction of} \\ \text{two matrices}\end{matrix}\right)$$

$$\mathbf{2A - 3B} = \begin{bmatrix} \mathbf{-4} & \mathbf{4} & \mathbf{-3} \\ \mathbf{-12} & \mathbf{8} & \mathbf{13} \\ \mathbf{8} & \mathbf{7} & \mathbf{15} \end{bmatrix} ✓$$

This is required solution.

☞ EXERCISE 3.2

Ex. 3.2.1 **W-15, 4 Marks.**

If $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 3 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$,

find (AB) · C.

☑ **Soln. :**

Matrix	Order	
$A = \begin{bmatrix} 2 & 10 \\ -1 & 32 \end{bmatrix}$	2×3	…(1)
$B = \begin{bmatrix} 1 & 3 \\ 3 & 0 \\ 0 & 1 \end{bmatrix}$	3×2	…(2)
$C = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$	2×2	…(3)

Same

$\therefore$ order of AB = 2×3 ; $3 \times 2 = 2 \times 2$

Order of AB

$\therefore$ AB exist

$$AB = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 2 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \end{matrix} \begin{bmatrix} 1 & 3 \\ 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$C_1 \quad C_2$

[From Equations (1) and (2)]

$$\therefore AB = \begin{bmatrix} R_1 C_1 & R_1 C_2 \\ R_2 C_1 & R_2 C_2 \end{bmatrix}$$

…Standard from …(4)

$R_1 C_1 = \underbrace{(2 \times 1)}_{2} + \underbrace{(1 \times 3)}_{3} + \underbrace{(0 \times 0)}_{0} = 2 + 3 + 0 = \mathbf{5}$

$R_1 C_2 = \underbrace{(2 \times 3)}_{6} + \underbrace{(1 \times 0)}_{0} + \underbrace{(0 \times 1)}_{0} = 6 + 0 + 0 = \mathbf{6}$

$R_2 C_1 = \underbrace{(-1 \times 1)}_{-1} + \underbrace{(3 \times 3)}_{9} + \underbrace{(2 \times 0)}_{0} = -1 + 9 + 0 = \mathbf{8}$

$R_2 C_2 = (-1 \times 3) + (-1 \times 0) + (2 \times 1) = -3 + 0 + 2 = \mathbf{-1}$

$$\therefore AB = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 8 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 6 \\ 8 & -1 \end{bmatrix} \text{ order of AB is } 2 \times 2 \qquad \text{…(5)}$$

▶ **Step II :** Now, order of (AB) · Same

C = 2×2 ; $2 \times 2 = 2 \times 2$

Order of (AB) · C

$\therefore$ (AB) · C exist

$$(AB) \cdot C = \begin{bmatrix} 5 & 6 \\ 8 & -1 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \end{matrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$$

$C_1 \quad C_2$

$$\therefore (AB) \cdot C = \begin{bmatrix} R_1 C_1 & R_1 C_2 \\ R_2 C_1 & R_2 C_2 \end{bmatrix}$$

…Standard from …(6)

$R_1 C_1 = \underbrace{(5 \times 1)}_{5} + \underbrace{(6 \times 3)}_{18} + = 5 + 18 = \mathbf{23}$

$R_1 C_2 = \underbrace{(5 \times 2)}_{10} + \underbrace{(6 \times -1)}_{-6} = 10 - 6 = \mathbf{4}$

$R_2 C_1 = \underbrace{(8 \times 1)}_{-1} + \underbrace{(-1 \times 3)}_{9} = 8 - 3 = \mathbf{5}$

$R_2 C_2 = \underbrace{\{8 \times 2\}}_{16} + \underbrace{(-1 \times (-1))}_{1} = 16 + 1 = \mathbf{17}$

Substituting all these values in Equation (6)

$$\therefore (AB) \cdot C = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix} = \begin{bmatrix} 23 & 4 \\ 5 & 17 \end{bmatrix}$$

$$\therefore\ (AB)\cdot C = \begin{bmatrix} 23 & 4 \\ 5 & 17 \end{bmatrix}$$ …Ans.

Ex. 3.2.2 **S-07, 4 Marks.**

If $A = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 4 & 2 & -5 \\ 1 & 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & -7 & 0 \\ -1 & 2 & 5 \\ 1 & 0 & 3 \end{bmatrix}$ then prove that (AB) C = A (BC)

Soln. : Step I : Given,

Matrix	Order	
$A = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$	2×2	…(1)
$B = \begin{bmatrix} 4 & 2 & -5 \\ 1 & 0 & 3 \end{bmatrix}$	2×3	…(2)
$C = \begin{bmatrix} 6 & -7 & 0 \\ -1 & 2 & 5 \\ 1 & 0 & 3 \end{bmatrix}$	3×3	…(3)

From Equation (1), and (2)

Order of AB = 2×2 ; 2×3 = 2×3 (same)

Order of AB

$$A \cdot B = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \end{matrix} \quad \begin{bmatrix} 4 & 2 & -5 \\ 1 & 0 & 3 \end{bmatrix}, \quad C_1\ C_2\ C_3$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{bmatrix}$$ …Standard form …(4)

Now,

$R_1C_1 = \underbrace{(1 \times 4)}_{4} + \underbrace{((-2) \times 1)}_{-2} = 4 + (-2) = \mathbf{2}$

$R_1C_2 = \underbrace{(1\times2)}_{2} + \underbrace{((-2)\times0)}_{0} = 2 + 0 = \mathbf{2}$

$R_1C_3 = \underbrace{(1\times(-5))}_{-5} + \underbrace{((-2)\times3)}_{-6} = (-5) + (-6) = \mathbf{-11}$

$R_2C_1 = \underbrace{((-3)\times4)}_{-12} + \underbrace{(1\times1)}_{1} = (-12) + 1 = \mathbf{-11}$

$R_2C_2 = \underbrace{((-3)\times2)}_{-6} + \underbrace{(1\times0)}_{0} = (-6) + 0 = \mathbf{-6}$

$R_2C_3 = \underbrace{((-3)\times(-5))}_{15} + \underbrace{(1\times3)}_{3} = 15 + 3 = \mathbf{18}$

Substituting all these values in Equation (4)

$$\therefore AB = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -11 \\ -11 & -6 & 18 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 2 & -11 \\ -11 & -6 & 18 \end{bmatrix}$$ …(3)

▶ Step II :

$$(AB)\,C = \begin{bmatrix} 2 & 2 & -11 \\ -11 & -6 & 18 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \end{matrix} \quad \begin{bmatrix} 6 & -7 & 0 \\ -1 & 2 & 5 \\ 1 & 0 & 3 \end{bmatrix}, \quad C_1\ C_2\ C_3$$ [From Equations (3) and (5)]

$$(AB)\cdot C = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{bmatrix}$$ …Standard form…(6)

Now,

$R_1C_1 = \underbrace{(2\times6)}_{12} + \underbrace{(2\times(-1))}_{-2} + \underbrace{((-11)\times1)}_{-11} = 12 - 2 - 11 = \mathbf{-1}$

$R_1C_2 = \underbrace{(2\times(-7))}_{-14} + \underbrace{(2\times2)}_{4} + \underbrace{((-11)\times0)}_{0} = -14 + 4 + 0 = \mathbf{-10}$

$R_1C_3 = \underbrace{(2\times0)}_{0} + \underbrace{(2\times5)}_{10} + \underbrace{((-11)\times3)}_{-33} = 0 + 10 - 33 = \mathbf{-23}$

$R_2C_1 = \underbrace{((-11)\times6)}_{-66} + \underbrace{((-6)\times(-1))}_{6} + \underbrace{(18\times1)}_{18} = -66 + 6 + 18$

$= \mathbf{-42}$

$R_2C_2 = \underbrace{((-11)\times(-7))}_{77} + \underbrace{((-6)\times2)}_{-12} + \underbrace{(18\times0)}_{0} = 77 - 12 + 0$

$= \mathbf{65}$

$R_2C_3 = \underbrace{((-11)\times0)}_{0} + \underbrace{((-6)\times5)}_{-30} + \underbrace{(18\times3)}_{54} = 0 - 30 + 54$

$= \mathbf{24}$

Substituting all these values in Equation (6)

$$(AB).C = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{bmatrix} = \begin{bmatrix} -1 & -10 & -23 \\ -42 & 65 & 24 \end{bmatrix}$$

$$\mathbf{(AB)C} = \begin{bmatrix} \mathbf{-1} & \mathbf{-10} & \mathbf{-23} \\ \mathbf{-42} & \mathbf{65} & \mathbf{24} \end{bmatrix}$$ …(7)

▶ Step III : **Now,**

$$BC = \begin{bmatrix} 4 & 2 & -5 \\ 1 & 0 & 3 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \end{matrix} \quad \begin{bmatrix} 6 & -7 & 0 \\ -1 & 2 & 5 \\ 1 & 0 & 3 \end{bmatrix}, \quad C_1\ C_2\ C_3$$ [From Equations (2) and (3)]

$$BC = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{bmatrix} \text{...Standard form} \quad \text{...(8)}$$

Now,

$R_1C_1 = \underbrace{(4\times 6)}_{24} + \underbrace{(2\times(-1))}_{-2} + \underbrace{((-5)\times 1)}_{-5} = 24 - 2 - 5 = \mathbf{17}$

$R_1C_2 = \underbrace{(4\times(-7))}_{-28} + \underbrace{(2\times 2)}_{4} + \underbrace{((-5)\times 0)}_{0} = -28 + 4 + 0 = \mathbf{-24}$

$R_1C_3 = \underbrace{(4\times 0)}_{0} + \underbrace{(2\times 5)}_{10} + \underbrace{((-5)\times 3)}_{-15} = 0 + 10 - 15 = \mathbf{-5}$

$R_2C_1 = \underbrace{(1\times 6)}_{6} + \underbrace{(0\times(-1))}_{0} + \underbrace{(3\times 1)}_{3} = 6 + 0 + 3 = \mathbf{9}$

$R_2C_2 = \underbrace{(1\times(-7))}_{-7} + \underbrace{(0\times 2)}_{0} + \underbrace{(3\times 0)}_{0} = -7 + 0 + 0 = \mathbf{-7}$

$R_2C_3 = \underbrace{(1\times 0)}_{0} + \underbrace{(0\times 5)}_{0} + \underbrace{(3\times 3)}_{9} = 0 + 0 + 9 = \mathbf{9}$

Substituting all these values in Equation (8)

$$BC = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{bmatrix} = \begin{bmatrix} 17 & -24 & -5 \\ 9 & -7 & 9 \end{bmatrix}$$

$$BC = \begin{bmatrix} 17 & -24 & -5 \\ 9 & -7 & 9 \end{bmatrix} \quad \text{...(9)}$$

- **Step IV :** Now,

$$A\cdot(BC) = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \begin{matrix} \to R_1 \\ \to R_2 \end{matrix} \begin{bmatrix} 17 & -24 & -5 \\ 9 & -7 & 9 \end{bmatrix}_{C_1\ C_2\ C_3}$$ [From Equations (1) and (9)]

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{bmatrix} \text{...Standard form} \quad \text{...(10)}$$

Now,

$R_1C_1 = \underbrace{(1\times 17)}_{17} + \underbrace{((-2)\times 9)}_{-18} = 17 - 18 = \mathbf{-1}$

$R_1C_2 = \underbrace{(1\times(-24))}_{-24} + \underbrace{((-2)\times(-7))}_{14} = -24 + 14 = \mathbf{-10}$

$R_1C_3 = \underbrace{(1\times(-5))}_{-5} + \underbrace{((-2)\times 9)}_{-18} = -5 - 18 = \mathbf{-23}$

$R_2C_1 = \underbrace{((-3)\times 17)}_{-51} + \underbrace{(1\times 9)}_{9} = -51 + 9 = \mathbf{-42}$

$R_2C_2 = \underbrace{((-3)\times(-24))}_{72} + \underbrace{(1\times(-7))}_{-7} = 72 - 7 = \mathbf{65}$

$R_2C_3 = \underbrace{((-3)\times(-5))}_{15} + \underbrace{(1\times 9)}_{9} = 15 + 9 = \mathbf{24}$

Substituting all these values in Equation (8)

$$A.(BC) = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{bmatrix} = \begin{bmatrix} -1 & -10 & -23 \\ -42 & 65 & 24 \end{bmatrix}$$

$$\mathbf{A\cdot(BC)} = \begin{bmatrix} \mathbf{-1} & \mathbf{-10} & \mathbf{-23} \\ \mathbf{-42} & \mathbf{65} & \mathbf{24} \end{bmatrix} \quad \text{...(11)}$$

- **Step V :** From Equations (7) and (11),

(AB) · C = A · (BC)✓ ...**Hence proved.**

Ex. 3.2.3 **W-06, W-07, W-10, 4 Marks**

If $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$ prove that $A^2 - 3A = 2I$

Where I is unit matrix of order two

☑ **Soln. :**

- **Step I :** Given matrix is,

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \text{order of matrix A is } 2\times 2 \quad \text{...(1)}$$

$$A^2 = A\times A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{matrix} \to R_1 \\ \to R_2 \end{matrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}_{C_1\ C_2}$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix} \text{...Standard form} \quad \text{...(2)}$$

Now,

$R_1C_1 = \underbrace{(2\times 2)}_{4} + \underbrace{(4\times 1)}_{4} = 4 + 4 = \mathbf{8}$

$R_1C_2 = \underbrace{(2\times 4)}_{8} + \underbrace{(4\times 1)}_{4} = 8 + 4 = \mathbf{12}$

$R_2C_1 = \underbrace{(1\times 2)}_{2} + \underbrace{(1\times 1)}_{1} = 2 + 1 = \mathbf{3}$

$R_2C_2 = \underbrace{(1\times 4)}_{4} + \underbrace{(1\times 1)}_{1} = 4 + 1 = \mathbf{5}$

Substituting all these values in Equation (2)

$$\therefore A^2 = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 3 & 5 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 8 & 12 \\ 3 & 5 \end{bmatrix} \qquad ...(3)$$

Step II :

$$A^2 - 3A = \begin{bmatrix} 8 & 12 \\ 3 & 5 \end{bmatrix} - 3\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$$

[From Equations (1) and (3)]

$$= \begin{bmatrix} 8 & 12 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 3\times 2 & 3\times 4 \\ 3\times 1 & 3\times 1 \end{bmatrix} \quad \text{(by scalar multiplication)}$$

$$= \begin{bmatrix} 8 & 12 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 12 \\ 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 8-6 & 12-12 \\ 3-3 & 5-3 \end{bmatrix} \quad \text{(Subtraction of two matrices)}$$

$$\mathbf{A^2 - 3A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}} \qquad ...(4)$$

Step III : Now,

$$2I = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{(I unit matrix of order } 2\times 2)$$

$$= \begin{bmatrix} 2\times 1 & 2\times 0 \\ 2\times 0 & 2\times 1 \end{bmatrix} \quad \text{(by scalar multiplication)}$$

$$\mathbf{2\,I = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}} \qquad ...(5)$$

From Equations (4) and (5),

$\mathbf{A^2 - 3A = 2I}$ ✓ **Hence proved.**

Ex. 3.2.4 **S-07, S-16, 4 Marks.**

If $\left\{ 3\begin{bmatrix} 3 & 1 \\ 4 & 0 \\ 3 & -3 \end{bmatrix} - 2\begin{bmatrix} 0 & 2 \\ -2 & 3 \\ -5 & 4 \end{bmatrix} \right\} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

then find x, y, z.

☑ **Soln. :** Given,

$$\left\{ 3\begin{bmatrix} 3 & 1 \\ 4 & 0 \\ 3 & -3 \end{bmatrix} - 2\begin{bmatrix} 0 & 2 \\ -2 & 3 \\ -5 & 4 \end{bmatrix} \right\} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 3\times 3 & 3\times 1 \\ 3\times 4 & 3\times 0 \\ 3\times 3 & 3\times(-3) \end{bmatrix} - \begin{bmatrix} 2\times 0 & 2\times 2 \\ 2\times(-2) & 2\times 3 \\ 2\times(-5) & 2\times 4 \end{bmatrix} \right\} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(By scalar multiplication)

$$\left\{ \begin{bmatrix} 9 & 3 \\ 12 & 0 \\ 9 & -9 \end{bmatrix} - \begin{bmatrix} 0 & 4 \\ -4 & 6 \\ -10 & 8 \end{bmatrix} \right\} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 9-0 & 3-4 \\ 12-(-4) & 0-6 \\ 9\ (-10) & -9-8 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \left(\text{subtraction of two matrices}\right)$$

$$\begin{matrix} R_1 \to \\ R_2 \to \\ R_3 \to \end{matrix} \begin{bmatrix} 9 & -1 \\ 16 & -6 \\ 19 & -17 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}_{\uparrow C_1} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} R_1\,C_1 \\ R_2\,C_1 \\ R_3\,C_1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{...Standard form} \qquad ...(1)$$

Now,

$$R_1\,C_1 = \underbrace{(9 \times (-1))}_{-9} + \underbrace{((-1)\times 2)}_{-2} = -9 - 2 = \mathbf{-11}$$

$$R_2C_1 = \underbrace{(16 \times (-1))}_{-16} + \underbrace{((-6)\times 2)}_{-12} = -16 - 12 = \mathbf{-28}$$

$$R_3C_1 = \underbrace{(19 \times (-1))}_{-19} + \underbrace{((-17)\times 2)}_{-34} = -19 - 34 = \mathbf{-53}$$

Substituting all these values in Equation (1)

$$\begin{bmatrix} -11 \\ -28 \\ -53 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} -11 \\ -28 \\ -53 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \left(\text{by Equality of two matrices}\right)$$

By Equating corresponding elements of matrices,

∴ $\mathbf{x = -11,\ y = -28,\ z = -53}$ ✓ **...Ans.**

Ex. 3.2.5 **S-09, 4 Marks.**

If $A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix}$ Find $A^2 - 3I$

☑ **Soln. :** Given matrix is,

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix} \quad \text{order of matrix A is } 3\times 3 \qquad ...(1)$$

$A^2 = A \cdot A$

$$= \begin{bmatrix} -1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix} \begin{bmatrix} -1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix} \quad \text{[From Equation (1)]}$$

$$\quad\quad\quad\quad\quad\quad\quad\quad \uparrow C_1 \ \uparrow C_2 \ \uparrow C_3$$

$$A^2 = \begin{bmatrix} R_1 C_1 & R_1 C_2 & R_1 C_3 \\ R_2 C_1 & R_2 C_2 & R_2 C_3 \\ R_3 C_1 & R_3 C_2 & R_3 C_3 \end{bmatrix} \text{…(Standard Form)} \quad \text{…(2)}$$

Now,

$R_1 C_1 = \underbrace{((-1) \times (-1))}_{1} + \underbrace{(2 \times 2)}_{4} + \underbrace{(3 \times 1)}_{3} = 1 + 4 + 3 = \mathbf{8}$

$R_1 C_2 = \underbrace{((-1)\times 2)}_{-2} + \underbrace{(2 \times 1)}_{2} + \underbrace{(3 \times (-1))}_{-3} = -2 + 2 - 3 = \mathbf{-3}$

$R_1 C_3 = \underbrace{((-1) \times 3)}_{-3} + \underbrace{(2 \times 2)}_{4} + \underbrace{(3 \times 3)}_{9} = -3 + 4 + 9 = \mathbf{10}$

$R_2 C_1 = \underbrace{(2 \times (-1))}_{-2} + \underbrace{(1 \times 2)}_{2} + \underbrace{(2 \times 1)}_{2} = -2 + 2 + 2 = \mathbf{2}$

$R_2 C_2 = \underbrace{(2 \times 2)}_{4} + \underbrace{(1 \times 1)}_{1} + \underbrace{(2 \times (-1))}_{-2} = 4 + 1 - 2 = \mathbf{3}$

$R_2 C_3 = \underbrace{(2 \times 3)}_{6} + \underbrace{(1 \times 2)}_{2} + \underbrace{(2 \times 3)}_{6} = 6 + 2 + 6 = \mathbf{14}$

$R_3 C_1 = \underbrace{(1 \times (-1))}_{-1} + \underbrace{((-1) \times 2)}_{-2} + \underbrace{(3 \times 1)}_{3} = -1 - 2 + 3 = \mathbf{0}$

$R_3 C_2 = \underbrace{(1 \times 2)}_{2} + \underbrace{((-1) \times 1)}_{-1} + \underbrace{(3 \times (-1))}_{-3} = 2 - 1 - 3 = \mathbf{-2}$

$R_3 C_3 = \underbrace{(1 \times 3)}_{3} + \underbrace{((-1)\times 2)}_{-2} + \underbrace{(3 \times 3)}_{9} = 3 - 2 + 9 = \mathbf{10}$

Substituting all these values in Equation (2)

$$\therefore A^2 = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix} = \begin{bmatrix} 8 & -3 & 10 \\ 2 & 3 & 14 \\ 0 & -2 & 10 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 8 & -3 & 10 \\ 2 & 3 & 14 \\ 0 & -2 & 10 \end{bmatrix} \quad \text{…(3)}$$

Now, $A^2 - 3I = \begin{bmatrix} 8 & -3 & 10 \\ 2 & 3 & 14 \\ 0 & -2 & 10 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(by Equation (3))

(I is unit matrix of order 3×3)

$$= \begin{bmatrix} 8 & -3 & 10 \\ 2 & 3 & 14 \\ 0 & -2 & 10 \end{bmatrix} - \begin{bmatrix} 3\times 1 & 3\times 0 & 3\times 0 \\ 3\times 0 & 3\times 1 & 3\times 0 \\ 3\times 0 & 3\times 0 & 3\times 1 \end{bmatrix}$$

(by scalar multiplication)

$$= \begin{bmatrix} 8 & -3 & 10 \\ 2 & 3 & 14 \\ 0 & -2 & 10 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 8-3 & -3-0 & 10-0 \\ 2-0 & 3-3 & 14-0 \\ 0-0 & -2-0 & 10-3 \end{bmatrix} \quad \left(\text{by subtraction of two matrix}\right)$$

$$\mathbf{A^2 - 3I} = \begin{bmatrix} \mathbf{5} & \mathbf{-3} & \mathbf{10} \\ \mathbf{2} & \mathbf{0} & \mathbf{14} \\ \mathbf{0} & \mathbf{-2} & \mathbf{7} \end{bmatrix} \checkmark \quad \textbf{…Ans.}$$

EXERCISE 3.3

Ex. 3.3.1 (W-11, 4 Marks)

If $A = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 6 \\ -3 & 4 \end{bmatrix}$

Find $(AB)^T$

Soln. : Given matrices are,

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix} \quad \text{…(1)}$$

$$B = \begin{bmatrix} 2 & 6 \\ -3 & 4 \end{bmatrix} \quad \text{…(2)}$$

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \end{matrix} \begin{bmatrix} 2 & 6 \\ -3 & 4 \end{bmatrix} \quad \text{[From Equations (1) and (2)]}$$

$$\quad\quad\quad\quad\quad \uparrow C_1 \ \uparrow C_2$$

$$AB = \begin{bmatrix} R_1 C_1 & R_1 C_2 \\ R_2 C_1 & R_2 C_2 \end{bmatrix} \text{…(Standard Form)} \quad \text{…(3)}$$

Now,

$R_1 C_1 = \underbrace{(1 \times 2)}_{2} + \underbrace{(2 \times (-3))}_{-6} = 2 - 6 = \mathbf{-4}$

$R_1 C_2 = \underbrace{(1 \times 6)}_{6} + \underbrace{(2 \times 4)}_{8} = 6 + 8 = \mathbf{14}$

$R_2 C_1 = \underbrace{(5 \times 2)}_{10} + \underbrace{(3 \times (-3))}_{-9} = 10 - 9 = \mathbf{1}$

$R_2\,C_2 = \underbrace{(5 \times 6)}_{30} + \underbrace{(3 \times 4)}_{12} = 30 + 12 = \mathbf{42}$

Substituting all these values in Equation (3)

$$\therefore AB = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix} = \begin{bmatrix} -4 & 14 \\ 1 & 42 \end{bmatrix}$$

$$AB = \begin{bmatrix} -4 & 14 \\ 1 & 42 \end{bmatrix} \quad \text{….(4)}$$

$$(AB)^T = \begin{bmatrix} -4 & 1 \\ 14 & 42 \end{bmatrix} ✓$$

[Traspose of matrix is interchange rows and columns]

This is required.

Ex. 3.3.2 (W-12, 4 Marks)

If $A = \begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix}$ verify that $(AB)^T = B^T A^T$

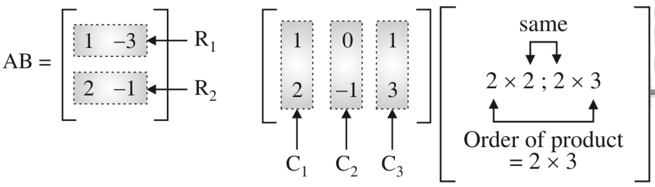
☑ **Soln. :** Given matrices are,

$$A = \begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix} \quad \text{…(1)}$$

A^T is Transpose of matrix A is a matrix by interchanging rows and columns of A.

$$\therefore A^T = \begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix} \text{ and } B^T = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 3 \end{bmatrix} \quad \text{…(2)}$$

▶ **Step I :** From Equation (1)



$$= \begin{bmatrix} R_1\,C_1 & R_1\,C_2 & R_1\,C_3 \\ R_2\,C_1 & R_2\,C_2 & R_2\,C_3 \end{bmatrix} \text{…(Standard Form)} \quad \text{…(3)}$$

Now,

$R_1\,C_1 = \underbrace{(1 \times 1)}_{1} + \underbrace{(-3 \times 2)}_{-6} = 1 - 6 = \mathbf{-5}$

$R_1\,C_2 = \underbrace{(1 \times 0)}_{0} + \underbrace{(-3 \times (-1))}_{3} = 0 + 3 = \mathbf{3}$

$R_1\,C_3 = \underbrace{(1 \times 1)}_{1} + \underbrace{((-3) \times 3)}_{-9} = 1 - 9 = \mathbf{-8}$

$R_2\,C_1 = \underbrace{(2 \times 1)}_{2} + \underbrace{((-1) \times 2)}_{-2} = 2 - 2 = \mathbf{0}$

$R_2\,C_2 = \underbrace{(2 \times 0)}_{0} + \underbrace{((-1) \times (-1))}_{1} = 0 + 1 = \mathbf{1}$

$R_2\,C_3 = \underbrace{(2 \times 1)}_{2} + \underbrace{((-1) \times 3)}_{-3} = 2 - 3 = \mathbf{-1}$

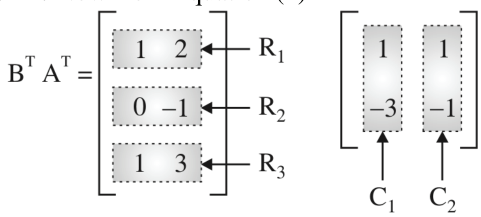
Substituting all these values in Equation (3)

$$\therefore AB = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{bmatrix} = \begin{bmatrix} -5 & 3 & -8 \\ 0 & 1 & -1 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} -5 & 3 & -8 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\therefore (AB)^T = \begin{bmatrix} -5 & 0 \\ 3 & 1 \\ -8 & -1 \end{bmatrix} \quad \text{…(4)}$$

▶ **Step II :** Now from Equation (2)



$$= \begin{bmatrix} R_1\,C_1 & R_1\,C_2 \\ R_2\,C_1 & R_2\,C_2 \\ R_3\,C_1 & R_3\,C_2 \end{bmatrix} \text{…(Standard Form)} \quad \text{…(5)}$$

Now,

$R_1\,C_1 = \underbrace{(1 \times 1)}_{1} + \underbrace{(2 \times (-3))}_{-6} = 1 - 6 = \mathbf{-5}$

$R_1\,C_2 = \underbrace{(1 \times 2)}_{2} + \underbrace{(2 \times (-1))}_{-2} = 2 - 2 = \mathbf{0}$

$R_2\,C_1 = \underbrace{(0 \times 1)}_{0} + \underbrace{(-1 \times (-3))}_{3} = 0 + 3 = \mathbf{3}$

$R_2\,C_2 = \underbrace{(0 \times 2)}_{0} + \underbrace{(-1 \times (-1))}_{1} = 0 + 1 = \mathbf{1}$

$R_3\,C_1 = \underbrace{(1 \times 1)}_{1} + \underbrace{(3 \times (-3))}_{-9} = 1 - 9 = \mathbf{-8}$

$R_3\,C_2 = \underbrace{(1 \times 2)}_{2} + \underbrace{(3 \times (-1))}_{-3} = 2 - 3 = \mathbf{-1}$

Substituting all these values in Equation (5)

$$\therefore B^T A^T = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \\ R_3C_1 & R_3C_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 3 & 1 \\ -8 & -1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} -5 & 0 \\ 3 & 1 \\ -8 & -1 \end{bmatrix} \quad \ldots(6)$$

Since R.H.S. of Equations (4) and (6) are equal

($\therefore$ L.H.S. also).

$\therefore$ **$(AB)^T = B^T A^T$** **Hence verified**

Ex. 3.3.3 (W-08, 4 Marks)

If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$

verify that $(AB)' = B'A'$

☑ **Soln. :** Given matrices are,

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix} \quad \ldots(1)$$

A' is a transpose of matrix A.

Transpose of matrix A is a matrix A' by interchanging rows and columns of A.

$$A' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}, \; B' = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 4 \\ 3 & 5 & 7 \end{bmatrix} \quad \ldots(2)$$

▸ **Step I :** Now, from Equation (1),

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$$
$$\qquad\qquad\qquad\qquad \uparrow C_1 \; \uparrow C_2 \; \uparrow C_3$$

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix} \ldots \left(\text{Standard Form}\right) \quad \ldots(3)$$

Now,

$R_1C_1 = \underbrace{(1\times1)}_{1} + \underbrace{(2\times1)}_{2} + \underbrace{(1\times2)}_{2} = 1+2+2 = \mathbf{5}$

$R_1C_2 = \underbrace{(1\times2)}_{2} + \underbrace{(2\times1)}_{2} + \underbrace{(1\times4)}_{4} = 2+2+4 = \mathbf{8}$

$R_1C_3 = \underbrace{(1\times3)}_{3} + \underbrace{(2\times5)}_{10} + \underbrace{(1\times7)}_{7} = 3+10+7 = \mathbf{20}$

$R_2C_1 = \underbrace{(0\times1)}_{0} + \underbrace{(2\times1)}_{2} + \underbrace{(3\times2)}_{6} = 0+2+6 = \mathbf{8}$

$R_2C_2 = \underbrace{(0\times2)}_{0} + \underbrace{(2\times1)}_{2} + \underbrace{(3\times4)}_{12} = 0+2+12 = \mathbf{14}$

$R_2C_3 = \underbrace{(0\times3)}_{0} + \underbrace{(2\times5)}_{10} + \underbrace{(3\times7)}_{21} = 0+10+21 = \mathbf{31}$

$R_3C_1 = \underbrace{(0\times1)}_{0} + \underbrace{(0\times1)}_{0} + \underbrace{(1\times2)}_{2} = 0+0+2 = \mathbf{2}$

$R_3C_2 = \underbrace{(0\times2)}_{0} + \underbrace{(0\times1)}_{0} + \underbrace{(1\times4)}_{4} = 0+0+4 = \mathbf{4}$

$R_3C_3 = \underbrace{(0\times3)}_{0} + \underbrace{(0\times5)}_{0} + \underbrace{(1\times7)}_{7} = 0+0+7 = \mathbf{7}$

Substituting all these values in Equation (3)

$$\therefore AB = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 20 \\ 8 & 14 & 31 \\ 2 & 4 & 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 8 & 20 \\ 8 & 14 & 31 \\ 2 & 4 & 7 \end{bmatrix} \quad \ldots(4)$$

$$(AB)' = \begin{bmatrix} 5 & 8 & 2 \\ 8 & 14 & 4 \\ 20 & 31 & 7 \end{bmatrix} \left[\text{By interchanging rows and columns of matrix (4)}\right] \quad \ldots(5)$$

▸ **Step II :** From Equation (2),

$$B'A' = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 4 \\ 3 & 5 & 7 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$
$$\qquad\qquad\qquad\qquad \uparrow C_1 \; \uparrow C_2 \; \uparrow C_3$$

$$= \begin{bmatrix} R_1 C_1 & R_1 C_2 & R_1 C_3 \\ R_2 C_1 & R_2 C_2 & R_2 C_3 \\ R_3 C_1 & R_3 C_2 & R_3 C_3 \end{bmatrix} \text{…(Standard Form)} \quad \text{…(6)}$$

Now,

$R_1 C_1 = \underbrace{(1 \times 1)}_{1} + \underbrace{(1 \times 2)}_{2} + \underbrace{(2 \times 1)}_{2} = 1 + 2 + 2 = \mathbf{5}$

$R_1 C_2 = \underbrace{(1 \times 0)}_{0} + \underbrace{(1 \times 2)}_{2} + \underbrace{(2 \times 3)}_{6} = 0 + 2 + 6 = \mathbf{8}$

$R_1 C_3 = \underbrace{(1 \times 0)}_{0} + \underbrace{(1 \times 0)}_{0} + \underbrace{(2 \times 1)}_{2} = 0 + 0 + 2 = \mathbf{2}$

$R_2 C_1 = \underbrace{(2 \times 1)}_{2} + \underbrace{(1 \times 2)}_{2} + \underbrace{(4 \times 1)}_{4} = 2 + 2 + 4 = \mathbf{8}$

$R_2 C_2 = \underbrace{(2 \times 0)}_{0} + \underbrace{(1 \times 2)}_{2} + \underbrace{(4 \times 3)}_{12} = 0 + 2 + 12 = \mathbf{14}$

$R_2 C_3 = \underbrace{(2 \times 0)}_{0} + \underbrace{(1 \times 0)}_{0} + \underbrace{(4 \times 1)}_{4} = 0 + 0 + 4 = \mathbf{4}$

$R_3 C_1 = \underbrace{(3 \times 1)}_{3} + \underbrace{(5 \times 2)}_{10} + \underbrace{(7 \times 1)}_{7} = 3 + 10 + 7 = \mathbf{20}$

$R_3 C_2 = \underbrace{(3 \times 0)}_{0} + \underbrace{(5 \times 2)}_{10} + \underbrace{(7 \times 3)}_{21} = 0 + 10 + 21 = \mathbf{31}$

$R_3 C_3 = \underbrace{(3 \times 0)}_{0} + \underbrace{(5 \times 0)}_{0} + \underbrace{(7 \times 1)}_{7} = 0 + 0 + 7 = \mathbf{7}$

Substituting all these values in Equation (6)

$$\therefore B'A' = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 2 \\ 8 & 14 & 4 \\ 20 & 31 & 7 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 5 & 8 & 2 \\ 8 & 14 & 4 \\ 20 & 31 & 7 \end{bmatrix} \quad \text{…(7)}$$

From Equations (5) and (7),

(AB)′ = B′ A′ **Hence verified.**

EXERCISE 3.4

Ex. 3.4.1 **(W-08, 4 Marks, S-11, 4 Marks)**

Find the inverse of matrix by adjoint method.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

OR Find the adjoint of matrix A.

Soln. : Given matrix is,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \text{Compare with} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We know,

$$A^{-1} = \frac{1}{|A|} \text{adj} \cdot A \quad \text{…(1)}$$

$$|A| = (1) \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - (2) \begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} + (3) \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$$

$= \mathbf{1}\ [\underbrace{(4)(6)}_{24} - \underbrace{(5)(5)}_{10}] - \mathbf{2}\ [\underbrace{(2)(6)}_{12} - \underbrace{(3)(5)}_{15}]$

$+ \mathbf{3}\ [\underbrace{(2)(5)}_{10} - \underbrace{(3)(4)}_{12}]$

$= \mathbf{1}\ [\underbrace{24 - 25}_{1}] - \mathbf{2}\ [\underbrace{12 - 15}_{-3}] + \mathbf{3}\ [\underbrace{10 - 12}_{-2}]$

$= \mathbf{1}\ (-1) - \mathbf{2}\ (-3) + \mathbf{3}\ (-2) = -1 + 6 - 6 = -1$

i.e. $|A| = -1$ …(2)

i.e. $|A| \neq 0 \quad \therefore A^{-1}$ exist

Minors of elements	Cofactors of elements
$a_{11}(=1) = \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix}$ $= (4)(6) - (5)(5)$ $= 24 - 25 = -1 = M_{11}$	$a_{11} = (-1)^{1+1} M_{11}$ $= (1)(-1)$ $= -1 = C_{11}$
$a_{12}(=2) = \begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix}$ $= (2)(6) - (3)(5)$ $= 12 - 15 = -3 = M_{12}$	$a_{12} = (-1)^{1+2} M_{12}$ $= (-1)(-3)$ $= 3 = C_{12}$
$a_{13}(=3) = \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$ $= (2)(5) - (3)(4)$ $= 10 - 12 = -2 = M_{13}$	$a_{13} = (-1)^{1+3} M_{13}$ $= (1)(-2)$ $= -2 = C_{13}$
$a_{21}(=2) = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$ $= (2)(6) - (5)(3)$ $= 12 - 15 = -3 = M_{21}$	$a_{21} = (-1)^{2+1} M_{21}$ $= (-1)(-3)$ $= 3 = C_{21}$
$a_{22}(=4) = \begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix}$ $= (1)(6) - (3)(3)$	$a_{22} = (-1)^{2+2} M_{22}$ $= (1)(-3)$ $= -3 = C_{22}$

Minors of elements	Cofactors of elements
$=6-9=-3=M_{22}$	
$a_{23}(=5) = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}$ $=(1)(5)-(3)(2)$ $=5-6=-1=M_{23}$	$a_{23} = (-1)^{2+3}M_{23}$ $= (-1)(-1)$ $= 1 = C_{23}$
$a_{31}(=3) = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$ $=(2)(5)-(4)(3)$ $=10-12=-2=M_{31}$	$a_{31} = (-1)^{3+1}M_{31}$ $= (1)(-2)$ $= -2 = C_{31=}$
$a_{32}(=5) = \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}$ $=(1)(5)-(2)(3)$ $=5-6=-1=M_{32}$	$a_{32} = (-1)^{3+2}M_{32}$ $= (-1)(-1)$ $= 1 = C_{32}$
$a_{33}(=6) = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$ $=(1)(4)-(2)(2)$ $=4-4=0=M_{33}$	$a_{33} = (-1)^{3+3}M_{33}$ $= (1)(0)$ $= 0 = C_{33}$

By these values,

$$\text{Matrix of cofactors} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

Adjoint of matrix A=Transpose of matrix of cofactors

$$= \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix} \quad \ldots(3)$$

From Equations (1), (2), (3),

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

$$\mathbf{A^{-1}} = \begin{bmatrix} \mathbf{1} & \mathbf{-3} & \mathbf{2} \\ \mathbf{-3} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{0} \end{bmatrix} ✓ \quad \textbf{…Ans.}$$

Ex. 3.4.2 **(W-06, 4 Marks)**

Find the inverse of the matrix $A = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Using adjoint matrix

☑ **Soln.:** :Given matrix is,

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} \text{ compare with } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We know,

$$A^{-1} = \frac{1}{|A|}\text{adj } A \quad \ldots(1)$$

$$|A| = \mathbf{1}\begin{vmatrix} 3 & -1 \\ -1 & 0 \end{vmatrix} - \mathbf{(-3)}\begin{vmatrix} -3 & -1 \\ 2 & 0 \end{vmatrix} + \mathbf{2}\begin{vmatrix} -3 & 3 \\ 2 & -1 \end{vmatrix}$$

$$= \mathbf{1}\,[\underbrace{(3)(0)}_{0} - \underbrace{(-1)(-1)}_{1}] + \mathbf{3}\,[\underbrace{(-3)(0)}_{0} - \underbrace{(2)(-1)}_{-2}] + \mathbf{2}\,[\underbrace{(-3)(-1)}_{3} - \underbrace{(2)(3)}_{6}]$$

$$= \mathbf{1}\,[\underbrace{0-1}_{-1}] + \mathbf{3}\,[\underbrace{0+2}_{2}] + \mathbf{2}\,[\underbrace{3-6}_{-3}] = \mathbf{1}(-1) + \mathbf{3}(2) + \mathbf{2}(-3)$$

$$|A| = -1+6-6 = -1 \quad \ldots(2)$$

$|A| \neq 0 \quad \therefore A^{-1}$ exist.

Minors of elements	Cofactors of elements
$a_{11}(=1) = \begin{vmatrix} 3 & -1 \\ -1 & 0 \end{vmatrix}$ $=(3)(0)-(-1)(-1)$ $=0-1=-1=M_{11}$	$a_{11} = (-1)^{1+1}M_{11}$ $= (1)(-1)$ $= -1 = C_{11}$
$a_{12}(=-3) = \begin{vmatrix} -3 & -1 \\ 2 & 0 \end{vmatrix}$ $=(-3)(0)-(2)(-1)$ $= 0+2=2=M_{12}$	$a_{12} = (-1)^{1+2}M_{12}$ $=(-1)(2)$ $= -2 = C_{12}$
$a_{13}(=2) = \begin{vmatrix} -3 & 3 \\ 2 & -1 \end{vmatrix}$ $=(-3)(-1)-(2)(3)$ $=3-6=-3=M_{13}$	$a_{13} = (-1)^{1+3}M_{13}$ $= (1)(-3)$ $= -3 = C_{13}$
$a_{21}(=-3) = \begin{vmatrix} -3 & 2 \\ -1 & 0 \end{vmatrix}$ $=(-3)(0)-(-1)(2)$ $=0+2=2=M_{21}$	$a_{21} = (-1)^{2+1}M_{21}$ $= (-1)(2)$ $= -2 = C_{21}$
$a_{22}(=3) = \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix}$ $=(1)(0)-(2)(2)$ $=0-4=-4=M_{22}$	$a_{22} = (-1)^{2+2}M_{22}$ $= (1)(-4)$ $= -4 = C_{22}$
$a_{23}(=-1) = \begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix}$ $=(1)(-1)-(2)(-3)$ $=-1+6=5=M_{23}$	$a_{23} = (-1)^{2+3}M_{23}$ $= (-1)(5)$ $= -5 = C_{23}$

Minors of elements	Cofactors of elements
$a_{31}(=2) = \begin{vmatrix} -3 & 2 \\ 3 & -1 \end{vmatrix}$ $=(-3)(-1)-(3)(2)$ $=3-6=-3=M_{31}$	$a_{31} = (-1)^{3+1} M_{31}$ $= (1)(-3)$ $= -3 = C_{31}$
$a_{32}(=-1) = \begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix}$ $=(1)(-1)-(-3)(2)$ $=-1+6=5=M_{32}$	$a_{32} = (-1)^{3+2} M_{32}$ $= (-1)(5)$ $= -5 = C_{32}$
$a_{33}(=0) = \begin{vmatrix} 1 & -3 \\ -3 & 3 \end{vmatrix}$ $=(1)(3)-(-3)(-3)$ $=3-9=-6=M_{33}$	$a_{33} = (-1)^{3+3} M_{33}$ $= (1)(-6)$ $= -6 = C_{33}$

By these values,

$$\text{Matrix of cofactors} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ -2 & -4 & -5 \\ -3 & -5 & -6 \end{bmatrix} \quad \ldots(3)$$

Adjoint of matrix A = transpose of matrix of cofactors

$$= \begin{bmatrix} -1 & -2 & -3 \\ -2 & -4 & -5 \\ -3 & -5 & -6 \end{bmatrix}$$

From Equations (1), (2) and (3)

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} -1 & -2 & -3 \\ -2 & -4 & -5 \\ -3 & -5 & -6 \end{bmatrix}$$

$$\mathbf{A^{-1}} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} & \mathbf{5} \\ \mathbf{3} & \mathbf{5} & \mathbf{6} \end{bmatrix} \checkmark \quad \textbf{...Ans.}$$

Ex. 3.4.3 : Find the inverse of matrix by adjoint method $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & -1 & 1 \end{bmatrix}$

☑ **Soln. :** Given matrix is,

$$Y = AX \quad \ldots(1)$$

Where, $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & -1 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & -1 & 1 \end{vmatrix}$$

$=1(3+4)-(2-4)+1(-2-3)=4$

$|A| \neq 0$ therefore matrix is non-singular and A^{-1} exist. …(2)

We have, $A^{-1} = \dfrac{1}{|A|}$ Adj. A

$$\text{Cofactors of each element of } A = \begin{bmatrix} 7 & 2 & -5 \\ -2 & 0 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\text{Adj. } A = \begin{bmatrix} 7 & -2 & 1 \\ 2 & 0 & -2 \\ -5 & 2 & 1 \end{bmatrix} \quad \left[\because \text{Adjoint of A is a transpose of cofactor matrix}\right]$$

$$\therefore \quad A^{-1} = \frac{1}{4} \begin{bmatrix} 7 & -2 & 1 \\ 2 & 0 & -2 \\ -5 & 2 & 1 \end{bmatrix}$$

Ex. 3.4.4 **S-15, 4 Marks.**

Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 4 \\ -1 & 2 & 3 \\ 1 & 4 & 1 \end{bmatrix}$ using adjoint method.

☑ **Soln. :** Given matrix is,

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 2 & 3 \\ 1 & 4 & 1 \end{bmatrix} \text{ compare with } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We know,

$$A^{-1} = \frac{1}{|A|} \text{ adj. A} \quad \ldots(1)$$

$$|A| = \mathbf{1} \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} - \mathbf{2} \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} + \mathbf{4} \begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix}$$

$$= \mathbf{1}\,[\underbrace{(2)(1)}_{2} - \underbrace{(4)(3)}_{12}] - \mathbf{2}\,[\underbrace{(-1)(1)}_{-1} - \underbrace{(1)(3)}_{3}] + \mathbf{4}\,[\underbrace{(-1)(4)}_{-4} - \underbrace{(1)(2)}_{2}]$$

$$= \mathbf{1}\,[\underbrace{2-12}_{-10}] - \mathbf{2}\,[\underbrace{-1-3}_{-4}] + \mathbf{4}\,[\underbrace{-4-2}_{-6}]$$

$$= \mathbf{1}(-10) - \mathbf{2}(-4) + \mathbf{4}(-6) = -10+8-24$$

$$|A| = -26 \quad \ldots(2)$$

i.e. $|A| \neq 0 \therefore A^{-1}$ exist

Minors of elements	Cofactors of element
$a_{11}(=1) = \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}$ $= (2)(1)-(4)(3)$ $= 2-12=-10=M_{11}$	$a_{11} = (-1)^{1+1} M_{11}$ $= (1)(-10)$ $= -10 = C_{11}$
$a_{12}(=2) = \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix}$ $= (-1)(1)-(1)(3)$	$a_{12} = (-1)^{1+2} M_{12}$ $= (-1)(-4)$ $= 4 = C_{12}$

Minors of elements	Cofactors of element
$= -1 - 3 = -4 = M_{12}$	
$a_{13} (= 4) = \begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix}$ $= (-1)(4) - (1)(2)$ $= -4 - 2 = -6 = M_{13}$	$a_{13} = (-1)^{1+3} M_{13}$ $= (1)(-6)$ $= -6 = C_{13}$
$a_{21} (= -1) = \begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix}$ $= (2)(1) - (4)(4)$ $= 2 - 16 = -14 = M_{21}$	$a_{21} = (-1)^{2+1} M_{21}$ $= (-1)(-14)$ $= 14 = C_{21}$
$a_{22} (= 2) = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix}$ $= (1)(1) - (1)(4)$ $= 1 - 4 = -3 = M_{22}$	$a_{22} = (-1)^{2+2} M_{22}$ $= (1)(-3)$ $= -3 = C_{22}$
$a_{23} (= 3) = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$ $= (1)(4) - (1)(2)$ $= 4 - 2 = 2 = M_{23}$	$a_{23} = (-1)^{2+3} M_{23}$ $= (-1)(2)$ $= -2 = C_{23}$
$a_{31} (= 1) = \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix}$ $= (2)(3) - (2)(4)$ $= 6 - 8 = -2 = M_{31}$	$a_{31} = (-1)^{3+1} M_{31}$ $= (1)(-2)$ $= -2 = C_{31}$
$a_{32} (= 4) = \begin{vmatrix} 1 & 4 \\ -1 & 3 \end{vmatrix}$ $= (1)(3) - (-1)(4)$ $= 3 + 4 = 7 = M_{32}$	$a_{32} = (-1)^{3+2} M_{32}$ $= (-1)(7)$ $= -7 = C_{32}$
$a_{33} (= 1) = \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix}$ $= (1)(2) - (-1)(2)$ $= 2 + 2 = 4 = M_{33}$	$a_{33} = (-1)^{3+3} M_{33}$ $= (1)(4)$ $= 4 = C_{33}$

$$\text{Matrix of cofactors} = \begin{bmatrix} C_{11} C_{12} C_{13} \\ C_{21} C_{22} C_{23} \\ C_{31} C_{32} C_{33} \end{bmatrix} = \begin{bmatrix} -10 & 4 & -6 \\ 14 & -3 & -2 \\ -2 & -7 & 4 \end{bmatrix}$$

$\therefore$ Adjoint of matrix A = Transpose of matrix of cofactors

$$= \begin{bmatrix} -10 & 14 & -2 \\ 4 & -3 & -7 \\ -6 & -2 & 4 \end{bmatrix} \qquad ...(3)$$

From Equations (1), (2), (3)

$$A^{-1} = \frac{1}{-26} \begin{bmatrix} -10 & 14 & -2 \\ 4 & -3 & -7 \\ -6 & -2 & 4 \end{bmatrix} \checkmark \qquad \textbf{...Ans.}$$

EXERCISE 3.5

Ex. 3.5.1 **W-12, 4 Marks.**

Using matrix inversion method solve the equation

$x + y + z = 5,\ x + y - z = 3,\ x - y = 2$

Soln. : Given equations are,

$x + y + z = 5$; $x + y - z = 3$; $x - y = 2$

These we can write in matrix form as,

$$AX = B$$

Where,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \text{ compare with } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Known as coefficient matrix

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} \qquad ...(1)$$

As, $AX = B$

$\therefore \quad X = A^{-1}B \qquad ...(2)$

We know,

$$A^{-1} = \frac{1}{|A|} \text{adj } A \qquad ...(3)$$

Since, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$

$= 1 \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

$= 1 [\underbrace{(1)(0)}_{0} - \underbrace{(-1)(-1)}_{1}] - 1 [\underbrace{(1)(0)}_{0} - \underbrace{(1)(-1)}_{-1}]$
$+ 1 [\underbrace{(1)(-1)}_{-1} - \underbrace{(1)(1)}_{1}]$

$= 1 [\underbrace{0 - 1}_{-1}] - 1 [\underbrace{0 + 1}_{1}] + 1 [\underbrace{-1 - 1}_{-2}] = 1(-1) - 1(1) + 1(-2)$

$= -1 - 1 - 2 = -4 \qquad ...(4)$

i.e. $= |A| \neq 0 \therefore A^{-1}$ exist

Minors of elements	Cofactors of elements
$a_{11} (= 1) = \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix}$ $= (1)(0) - (-1)(-1)$ $= 0 - 1 = -1 = M_{11}$	$a_{11} = (-1)^{1+1} M_{11}$ $= (1)(-1)$ $= -1 = C_{11}$
$a_{12} (= 1) = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}$ $= (1)(0) - (1)(-1)$	$a_{12} = (-1)^{1+2} M_{12}$ $= (-1)(1)$ $= -1 = C_{12}$

Minors of elements	Cofactors of elements
$=0 + 1 = 1 = M_{12}$	
$a_{13}(=1) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$ $=(1)(-1) - (1)(1)$ $= -1 - 1 = -2 = M_{13}$	$a_{13} = (-1)^{1+3} M_{13}$ $= (1)(-2)$ $= -2 = C_{13}$
$a_{21}(= 1) = \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix}$ $=(1)(0) - (-1)(1)$ $=0 + 1 = 1 = M_{21}$	$a_{21} = (-1)^{2+1} M_{21}$ $= (-1)(1)$ $= -1 = C_{21}$
$a_{22}(= 1) = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$ $=(1)(0) - (1)(1)$ $=0 - 1 = -1 = M_{22}$	$a_{22} = (-1)^{2+2} M_{22}$ $= (1)(-1)$ $= -1 = C_{22}$
$a_{23}(= -1) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$ $=(1)(-1) - (1)(1)$ $= -1 - 1 = -2 = M_{23}$	$a_{23} = (-1)^{2+3} M_{23}$ $= (-1)(-2)$ $= 2 = C_{23}$
$a_{31}(=) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$ $= (1)(-1) - (1)(1)$ $= -1 - 1 = -2 = M_{31}$	$a_{31} = (-1)^{3+1} M_{31}$ $= (1)(-2)$ $= -2 = C_{31}$
$a_{32}(= -1) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$ $=(1)(-1) - (1)(1)$ $= -1 - 1 = -2 = M_{32}$	$a_{32} = (-1)^{3+2} M_{32}$ $= (-1)(-2)$ $= 2 = C_{32}$
$a_{33}(= 0) = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$ $=(1)(1) - (1)(1)$ $=1 - 1 = 0 = M_{33}$	$a_{33} = (-1)^{3+3} M_{33}$ $= (1)(0)$ $= 0 = C_{33}$

By these values,

$$\text{Matrix of cofactors} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} -1 & -1 & -2 \\ -1 & -1 & 2 \\ -2 & 2 & 0 \end{bmatrix}$$

Adjoint of Matrix A=Transpose of matrix of cofactors

$$= \begin{bmatrix} -1 & -1 & -2 \\ -1 & -1 & 2 \\ -2 & 2 & 0 \end{bmatrix} \qquad \ldots(5)$$

From Equations (3), (4) and (5)

$$\therefore A^{-1} = \frac{1}{-4}\begin{bmatrix} -1 & -1 & -2 \\ -1 & -1 & 2 \\ -2 & 2 & 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

Substitute these values in Equation (2)

$$X = \frac{1}{4}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ 2 & -2 & 0 \end{bmatrix} B$$

$$X = \frac{1}{4}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ 2 & -2 & 0 \end{bmatrix}\begin{matrix} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{matrix}\begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

$$\uparrow$$
$$C_1$$

$$= \frac{1}{4}\begin{bmatrix} (1 \times 5) + (1 \times 3) + (2 \times 2) \\ (1 \times 5) + (1 \times 3) + (-2 \times 2) \\ (2 \times 5) + (-2 \times 3) + (0 \times 2) \end{bmatrix}$$

$$= \begin{bmatrix} R_1 C_1 \\ R_2 C_1 \\ R_3 C_1 \end{bmatrix} \qquad [\because \text{Multiplication of two matrices}]$$

$$X = \frac{1}{4}\begin{bmatrix} 5 + 3 + 4 \\ 5 + 3 - 4 \\ 10 - 6 + 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 12 \\ 4 \\ 4 \end{bmatrix} \quad \because \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{1}{4} \times 12 \\ \frac{1}{4} \times 4 \\ \frac{1}{4} \times 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

By Equating corresponding elements of both sides,

$\therefore$ **x = 3, y = 1 , z = 1✓** **…Ans.**

Ex. 3.5.2 W-10, 4 Marks.

Solve by matrix method

$2x + 3y - z = -3$, $5x + y + 3z = 10$, $4x + 3y - 2z = -3$

☑ **Soln. :** Given equations are,

$$2x + 3y - z = -3$$
$$5x + y + 3z = 10$$
$$4x + 3y - 2z = -3$$

These we can write in matrix form as,

$$AX = B$$

Where,

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 1 & 3 \\ 4 & 3 & -2 \end{bmatrix} \text{ compare with } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

known as coefficient matrix

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 \\ 10 \\ -3 \end{bmatrix} \quad \ldots(1)$$

As, $AX = B$

$\therefore \quad X = A^{-1}B \quad \ldots(2)$

We know,

$$A^{-1} = \frac{1}{|A|} \text{adj } A \quad \ldots(3)$$

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 5 & 1 & 3 \\ 4 & 3 & -2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} - 3\begin{vmatrix} 5 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} 5 & 1 \\ 4 & 3 \end{vmatrix}$$

$$= 2\,[\underbrace{(1)(-2)}_{-2} - \underbrace{(3)(3)}_{9}] - 3\,[\underbrace{(5)(-2)}_{-10} - \underbrace{(4)(3)}_{12}] - 1\,[\underbrace{(5)(3)}_{15} - \underbrace{(4)(1)}_{4}]$$

$$= 2\,[\underbrace{-2-9}_{-11}] - 3\,[\underbrace{-10-12}_{-22}] - 1\,[\underbrace{15-4}_{11}]$$

$$= 2(-11) - 3(-22) - 1(11) = -22 + 66 - 11$$

$$|A| = 33 \quad \ldots(4)$$

Minors of elements	Cofactors of elements
$a_{11} (= 2) = \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix}$ $= (1)(-2) - (3)(3)$ $= -2 - 9 = -11 = M_{11}$	$a_{11} = (-1)^{1+1} M_{11} = (1)(-11)$ $= -11 = C_{11}$
$a_{12} (= 3) = \begin{vmatrix} 5 & 3 \\ 4 & -2 \end{vmatrix}$ $= (5)(-2) - (4)(3)$ $= -10 - 12 = -22 = M_{12}$	$a_{12} = (-1)^{1+2} M_{12} = (-1)(-22)$ $= 22 = C_{12}$
$a_{13} (= -1) = \begin{vmatrix} 5 & 1 \\ 4 & 3 \end{vmatrix}$ $= (5)(3) - (4)(1)$ $= 15 - 4 = 11 = M_{13}$	$a_{13} = (-1)^{1+3} M_{13} = (1)(11)$ $= 11 = C_{13}$
$a_{21} (= 5) = \begin{vmatrix} 3 & -1 \\ 3 & -2 \end{vmatrix}$ $= (3)(-2) - (3)(-1)$ $= -6 + 3 = -3 = M_{21}$	$a_{21} = (-1)^{2+1} M_{21} = (-1)(-3)$ $= 3 = C_{21}$
$a_{22} (= 1) = \begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix}$ $= (2)(-2) - (4)(-1)$ $= -4 + 4 = 0 = M_{22}$	$a_{22} = (-1)^{2+2} M_{22} = (1)(0)$ $= 0 = C_{22}$

Minors of elements	Cofactors of elements
$a_{23} (= 3) = \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix}$ $= (2)(3) - (4)(3)$ $= 6 - 12 = -6 = M_{23}$	$a_{23} = (-1)^{2+3} M_{23} = (-1)(-6)$ $= 6 = C_{23}$
$a_{31} (= 4) = \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix}$ $= (3)(3) - (1)(-1)$ $= 9 + 1 = 10 = M_{31}$	$a_{31} = (-1)^{3+1} M_{31} = (1)(10)$ $= 10 = C_{31}$
$a_{32} (= 3) = \begin{vmatrix} 2 & -1 \\ 5 & 3 \end{vmatrix}$ $= (2)(3) - (5)(-1)$ $= 6 + 5 = 11 = M_{32}$	$a_{32} = (-1)^{3+2} M_{32} = (-1)(11)$ $= -11 = C_{32}$
$a_{33} (= -2) = \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix}$ $= (2)(1) - (5)(3)$ $= 2 - 15 = -13 = M_{33}$	$a_{33} = (-1)^{3+3} M_{33} = (1)(-13)$ $= -13 = C_{33}$

By these values,

$$\text{Matrix of cofactors} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} -11 & 22 & 11 \\ 3 & 0 & 6 \\ 10 & -11 & -13 \end{bmatrix}$$

Adjoint of matrix A = Transpose of matrix of cofactors

$$\therefore \quad \text{Adj } A = \begin{bmatrix} -11 & 3 & 10 \\ 22 & 0 & -11 \\ 11 & 6 & -13 \end{bmatrix} \quad \ldots(5)$$

Substitute values from Equations (4) and (5) in Equation (3)

$$\therefore \quad A^{-1} = \frac{1}{33}\begin{bmatrix} -11 & 3 & 10 \\ 22 & 0 & -11 \\ 11 & 6 & -13 \end{bmatrix}$$

Substitute this value in Equation (2)

$$X = \frac{1}{33}\begin{bmatrix} -11 & 3 & 10 \\ 22 & 0 & -11 \\ 11 & 6 & -13 \end{bmatrix} B$$

$$= \frac{1}{33}\begin{bmatrix} 11 & 3 & 10 \\ 22 & 0 & -11 \\ 11 & 6 & -13 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix} \begin{bmatrix} -3 \\ 10 \\ -3 \end{bmatrix}$$

$\uparrow C_1$

$$= \begin{bmatrix} R_1 C_1 \\ R_2 C_1 \\ R_3 C_1 \end{bmatrix} \quad [\because \text{ Multiplication of two matrices}]$$

$$X = \frac{1}{33}\begin{bmatrix} (-11 \times (-3)) + (3 \times 10) + (10 \times (-3)) \\ (22 \times (-3)) + (0 \times 10) + (-11 \times (-3)) \\ (11 \times (-3)) + (6 \times 10) + (-13 \times (-3)) \end{bmatrix}$$

$$= \frac{1}{33}\begin{bmatrix} 33+30-30 \\ -66+0+33 \\ -33+60+39 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{33}\begin{bmatrix} 33 \\ -33 \\ 66 \end{bmatrix} \qquad \left\{ \because X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\}$$

$$X = \begin{bmatrix} \frac{1}{33}\times 33 \\ \frac{1}{33}\times(-33) \\ \frac{1}{33}\times 66 \end{bmatrix} \qquad \text{[by scalar multiplication]}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

By Equating corresponding elements of both sides,

$\therefore$ **x = 1, y = – 1 , z = 2** ✓ **…Ans.**

Ex. 3.5.3 S-09, S-11, 4 Marks.

Solve by matrix method the set of equations : $x + y + z = 2,\ y + z = 1,\ z + x = 3$

☑ **Soln. :**

Given equations are,

$$x + y + z = 2\ ; \qquad y + z = 1$$
$$z + x = 3$$

Rewrite equations as,

$$x + y + z = 2\ ; \qquad 0x + y + z = 1$$
$$x + 0y + z = 3$$

These we can write in matrix form as,

$$AX = B$$

$$\text{Where, } A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \text{ compare with } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

known as coefficient matrix

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \qquad \text{…(1)}$$

$$\text{As, } AX = B$$

$$\therefore \quad X = A^{-1}B \qquad \text{…(2)}$$

$$\text{We know, } A^{-1} = \frac{1}{|A|}\text{ adj } A \qquad \text{…(3)}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \mathbf{1}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - \mathbf{1}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + \mathbf{1}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= \mathbf{1}\,[\underbrace{(1)(1)}_{1} - \underbrace{(0)(1)}_{0}] - \mathbf{1}\,[\underbrace{(0)(1)}_{0} - \underbrace{(1)(1)}_{1}]$$
$$+ \mathbf{1}\,[\underbrace{(0)(0)}_{0} - \underbrace{(1)(1)}_{1}]$$

$$= \mathbf{1}\,\underbrace{[1-0]}_{1} - \mathbf{1}\,\underbrace{[0-1]}_{-1} + \mathbf{1}\,\underbrace{[0-1]}_{-1} = \mathbf{1}\,(1) - \mathbf{1}\,(-1) + \mathbf{1}\,(-1)$$

$$= 1 + 1 - 1 = 1 \qquad \text{…(4)}$$

Minors of elements	Cofactors of elements
$a_{11}\ (=1) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$ $= (1)(1) - (0)(1)$ $= 1 - 0 = 1 = M_{11}$	$a_{11} = (-1)^{1+1} M_{11}$ $= (1)(-1)$ $= 1 = C_{11}$
$a_{12}\ (=1) = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$ $= (0)(1) - (1)(1)$ $= 0 - 1 = -1 = M_{12}$	$a_{12} = (-1)^{1+2} M_{12}$ $= (-1)(-1)$ $= 1 = C_{12}$
$a_{13}\ (=1) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ $= (0)(0) - (1)(1)$ $= 0 - 1 = -1 = M_{13}$	$a_{13} = (-1)^{1+3} M_{13}$ $= (1)(-1)$ $= -1 = C_{13}$
$a_{21}\ (=0) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$ $= (1)(1) - (0)(1)$ $= 1 - 0 = 1 = M_{21}$	$a_{21} = (-1)^{2+1} M_{21}$ $= (-1)(1)$ $= -1 = C_{21}$
$a_{22}\ (=1) = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$ $= (1)(1) - (1)(1)$ $= 1 - 1 = 0 = M_{22}$	$a_{22} = (-1)^{2+2} M_{22}$ $= (1)(0)$ $= 0 = C_{22}$
$a_{23}\ (=1) = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$ $= (1)(0) - (1)(1)$ $= 0 - 1 = -1 = M_{23}$	$a_{23} = (-1)^{2+3} M_{23}$ $= (-1)(-1)$ $= 1 = C_{23}$
$a_{31}\ (=1) = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$ $= (1)(1) - (1)(1)$ $= 1 - 1 = 0 = M_{31}$	$a_{31} = (-1)^{3+1} M_{31}$ $= (1)(0)$ $= 0 = C_{31}$
$a_{32}\ (=0) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$ $= (1)(1) - (0)(1)$ $= 1 - 0 = 1 = M_{32}$	$a_{32} = (-1)^{3+2} M_{32}$ $= (-1)(1)$ $= -1 = C_{32}$
$a_{33}\ (=1) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$ $= (1)(1) - (0)(1)$ $= 1 - 0 = 1 = M_{33}$	$a_{33} = (-1)^{3+3} M_{33}$ $= (1)(1)$ $= 1 = C_{33}$

By these values,

$$\text{Matrix of cofactors} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Adjoint of matrix A=Transpose of matrix of cofactors

$$= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} \qquad \ldots(5)$$

Substitute values from Equations (4) and (5) in Equation (3)

$$\therefore \quad A^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

Substitute this value in Equation (2)

$$X = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} B \qquad \{X = A^{-1} B\}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\uparrow C_1$$

$$= \begin{bmatrix} R_1 C_1 \\ R_2 C_1 \\ R_3 C_1 \end{bmatrix} \qquad [\because \text{ Multiplication of two matrix}]$$

$$X = \begin{bmatrix} (1 \times 2) + (-1 \times 1) + (0 \times 3) \\ (1 \times 2) + (0 \times 1) + (-1 \times 3) \\ (-1 \times 2) + (1 \times 1) + (1 \times 3) \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 1 + 0 \\ 2 + 0 - 3 \\ -2 + 1 + 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \left\{ \because X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\}$$

By Equating corresponding elements of matrices of both sides,

$\therefore$ **x = 1, y = – 1 , z = 2✓** …**Ans.**

Ex. 3.5.4 W-08, 4 Marks.

Using matrix method, Solve the simultaneous equations

$x + y + z = 6, \quad x - y + 2z = 5, \quad 2x + y - z = 1$

☑ **Soln. :** Given equations are,

$x + y + z = 6; \; x - y + 2z = 5 \; ; 2x + y - z = 1$

These we can write in matrix form as,

$$AX = B$$

$$\text{Where,} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\text{Compare with} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Known as coefficient matrix

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; \quad B = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} \qquad \ldots(1)$$

$$\text{As, } AX = B \qquad \therefore \quad X = A^{-1}B \qquad \ldots(2)$$

We know,

$$A^{-1} = \frac{1}{|A|} \text{ Adj } A \qquad \ldots(3)$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= 1\,[\underbrace{(-1)(-1)}_{1} - \underbrace{(1)(2)}_{2}] - 1\,[\underbrace{(1)(-1)}_{-1} - \underbrace{(2)(2)}_{4}] + 1\,[\underbrace{(1)(1)}_{1} - \underbrace{(2)(-1)}_{-2}]$$

$$= 1\,[\underbrace{1 - 2}_{-1}] - 1\,[\underbrace{-1 - 4}_{-5}] + 1\,[\underbrace{1 + 2}_{3}] = 1(-1) - 1(-5) + 1(3)$$

$$= -1 + 5 + 3 = 7 \qquad \ldots(4)$$

Minors of elements :	Cofactors of elements :
$a_{11} (= 1) = \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}$ $= (-1)(-1) - (1)(2)$ $= 1 - 2 = -1 = M_{11}$	$a_{11} = (-1)^{1+1} M_{11} = (1)(-1)$ $= -1 = C_{11}$
$a_{12} (= 1) = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$ $= (1)(-1) - (2)(2)$ $= -1 - 4 = -5 = M_{12}$	$a_{12} = (-1)^{1+2} M_{12} = (-1)(-5)$ $= 5 = C_{12}$
$a_{13} (= 1) = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$ $= (1)(1) - (2)(-1)$	$a_{13} = (-1)^{1+3} M_{13} = (1)(3)$ $= 3 = C_{13}$

Minors of elements :	Cofactors of elements :
$= -1 + 2 = 3 = M_{13}$	
$a_{21}\ (=1) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$ $= (1)(-1) - (1)(1)$ $= -1 - 1 = -2 = M_{21}$	$a_{21} = (-1)^{2+1} M_{21} = (-1)(-2)$ $= 2 = C_{21}$
$a_{22}\ (=-1) = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$ $= (1)(-1) - (2)(1)$ $= -1 - 2 = -3 = M_{22}$	$a_{22} = (-1)^{2+2} M_{22} = (1)(-3)$ $= -3 = C_{22}$
$a_{23}\ (=2) = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$ $= (1)(1) - (2)(1)$ $= 1 - 2 = -1 = M_{23}$	$a_{23} = (-1)^{2+3} M_{23} = (-1)(-1)$ $= 1 = C_{23}$
$a_{31}\ (=2) = \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$ $= (1)(2) - (-1)(1)$ $= 2 + 1 = 3 = M_{31}$	$a_{31} = (-1)^{3+1} M_{31} = (1)(3)$ $= 3 = C_{31}$
$a_{32}\ (=1) = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$ $= (1)(2) - (1)(1)$ $= 2 - 1 = 1 = M_{32}$	$a_{32} = (-1)^{3+2} M_{32} = (-1)(1)$ $= -1 = C_{32}$
$a_{33}\ (=-1) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$ $= (1)(-1) - (1)(1)$ $= -1 - 1 = -2 = M_{33}$	$a_{33} = (-1)^{3+3} M_{33} = (1)(-2)$ $= -2 = C_{33}$

By these values,

$$\text{Matrix of cofactors} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} -1 & 5 & 3 \\ 2 & -3 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

Adjoint of matrix A=Transpose of matrix of cofactors

$$= \begin{bmatrix} -1 & 2 & 3 \\ 5 & -3 & -1 \\ 3 & 1 & -2 \end{bmatrix}$$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} -1 & 2 & 3 \\ 5 & -3 & -1 \\ 3 & 1 & -2 \end{bmatrix} \qquad \ldots(5)$$

Substitute values from Equations (4) and (5) in Equation (3)

$$\therefore \quad A^{-1} = \frac{1}{7}\begin{bmatrix} -1 & 2 & 3 \\ 5 & -3 & -1 \\ 3 & 1 & -2 \end{bmatrix}$$

Using this in Equation (2)

$$X = \frac{1}{7}\begin{bmatrix} -1 & 2 & 3 \\ 5 & -3 & -1 \\ 3 & 1 & -2 \end{bmatrix}$$

$$B = \frac{1}{7}\begin{bmatrix} -1 & 2 & 3 \\ 5 & -3 & -1 \\ 3 & 1 & -2 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix} \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$$
$$\uparrow C_1$$

$$= \begin{bmatrix} R_1\,C_1 \\ R_2\,C_1 \\ R_3\,C_1 \end{bmatrix} \qquad [\because \text{ Multiplication of two matrix}]$$

$$X = \frac{1}{7}\begin{bmatrix} (-1\times 6) + (2\times 5) + (3\times 1) \\ (5\times 6) + (-3\times 5) + (-1\times 1) \\ (3\times 6) + (1\times 5) + (-2\times 1) \end{bmatrix}$$

$$X = \frac{1}{7}\begin{bmatrix} -6 + 10 + 3 \\ 30 - 15 - 1 \\ 18 + 5 - 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ 14 \\ 21 \end{bmatrix} \qquad \left\{ \because \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\}$$

$$X = \begin{bmatrix} \frac{1}{7}\times 7 \\ \frac{1}{7}\times 14 \\ \frac{1}{7}\times 21 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \left\{ \because \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\}$$

By Equating corresponding elements of both sides,

$\therefore$ **x = 1, y = 2, z = 3**✓ **…Ans.**

Ex. 3.5.5 S-08, S-10,S-12, S-13, 4 Marks, Q. 6(c), S-22, Q. 6(a), W-17, 6 Marks.

Using matrix inversion method solve the equation $x + y + z = 3$, $x + 2y + 3z = 4$, $x + 4y + 9z = 6$

☑ **Soln. :**

Given equations are,

$x + y + z = 3$; $x + 2y + 3z = 4$; $x + 4y + 9z = 6$

These we can write in the matrix form as,

$$AX = B$$

$$\text{Where,} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$\text{Compare with} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

known as coefficient matrix

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \; ; \quad B = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \qquad \ldots(1)$$

As, $AX = B$

$\therefore \quad X = A^{-1}B \qquad \ldots(2)$

We know,

$$A^{-1} = \frac{1}{|A|} \text{Adj } A \qquad \ldots(3)$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= (1)\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} - (1)\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} + (1)\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} \text{ s}$$

$$= 1\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$$

$$= 1\,[\underbrace{(2)(9)}_{18} - \underbrace{(4)(3)}_{12}] - 1\,[\underbrace{(1)(9)}_{9} - \underbrace{(1)(3)}_{3}] + 1\,[\underbrace{(1)(4)}_{4} - \underbrace{(1)(2)}_{2}]$$

$$= 1\,[\underbrace{18 - 12}_{6}] - 1\,[\underbrace{9 - 3}_{6}] + 1\,[\underbrace{4 - 2}_{2}] = 1\,(6) - 1\,(6) + 1\,(2)$$

$$= 6 - 6 + 2 = 2 \qquad \ldots(4)$$

Minors of elements	Cofactors of elements
$a_{11} (= 1) = \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix}$ $= (2)(9) - (4)(3)$ $= 18 - 12 = 6 = M_{11}$	$a_{11} = (-1)^{1+1} M_{11} = (1)(6)$ $= 6 = C_{11}$
$a_{12} (= 1) = \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix}$ $= (1)(9) - (1)(3)$ $= 9 - 3 = 6 = M_{12}$	$a_{12} = (-1)^{1+2} M_{12} = (-1)(6)$ $= -6 = C_{12}$
$a_{13} (= 1) = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$ $= (1)(4) - (1)(2)$ $= 4 - 2 = 2 = M_{13}$	$a_{13} = (-1)^{1+3} M_{13} = (1)(2)$ $= 2 = C_{13}$
$a_{21} (= 1) = \begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix}$ $= (1)(9) - (4)(1)$ $= 9 - 4 = 5 = M_{21}$	$a_{21} = (-1)^{2+1} M_{21} = (-1)(5)$ $= -5 = C_{21}$
$a_{22} (= 2) = \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix}$ $= (1)(9) - (1)(1)$ $= 9 - 1 = 8 = M_{22}$	$a_{22} = (-1)^{2+2} M_{22} = (1)(8)$ $= 8 = C_{22}$

Minors of elements	Cofactors of elements
$a_{23} (= 3) = \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}$ $= (1)(4) - (1)(1)$ $= 4 - 1 = 3 = M_{23}$	$a_{23} = (-1)^{2+3} M_{23} = (-1)(3)$ $= -3 = C_{23}$
$a_{31} (= 1) = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$ $= (1)(3) - (2)(1)$ $= 3 - 2 = 1 = M_{31}$	$a_{31} = (-1)^{3+1} M_{31} = (1)(1)$ $= 1 = C_{31}$
$a_{32} (= 4) = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}$ $= (1)(3) - (1)(1)$ $= 3 - 1 = 2 = M_{32}$	$a_{32} = (-1)^{3+2} M_{32} = (-1)(2)$ $= -2 = C_{32}$
$a_{33} (= 9) = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$ $= (1)(2) - (1)(1)$ $= 2 - 1 = 1 = M_{33}$	$a_{33} = (-1)^{3+3} M_{33} = (1)(1)$ $= 1 = C_{33}$

By these values,

$$\text{Matrix of cofactors} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

Adjoint of matrix A=Transpose of matrix of cofactors

$$= \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix} \qquad \ldots(5)$$

Substitute values from Equations (4) and (5) in Equation (3)

$$\therefore \quad A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

Substitute this value in Equation (2)

$$X = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix} B$$

$$= \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix} \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$\uparrow C_1$ [B From Equations (1)]

$$= \begin{bmatrix} R_1 C_1 \\ R_2 C_2 \\ R_3 C_3 \end{bmatrix}$$

[$\because$ by Multiplication of two matrices]

$$= \frac{1}{2}\begin{bmatrix} (6\times 3)+(-5\times 4)+(1\times 6) \\ (-6\times 3)+(8\times 4)+(-2\times 6) \\ (2\times 3)+(-3\times 4)+(1\times 6) \end{bmatrix}$$

$$X = \frac{1}{2}\begin{bmatrix} 18-20+6 \\ -18+32-12 \\ 6-12+6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} \qquad \left\{ \because \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\times 4 \\ \frac{1}{2}\times 2 \\ \frac{1}{2}\times 0 \end{bmatrix} \text{(by scalar multiplication)}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

By equating corresponding elements of matrices of both sides,

$\therefore$ **x = 2, y = 1 , z = 0**✓ **…Ans.**

Chapter Ends...

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Chapter 4 : PARTIAL FRACTION

Exercise 4.1

Ex. 4.1.1 (S-11, S-12, 2 Marks)

Resolve into partial fraction : $\frac{1}{x^2 - x}$

☑ **Soln. :** First find out all possible factors of denominator

Since, $\frac{1}{x^2 - x} = \frac{1}{x(x-1)}$

Consider, $$\frac{1}{x^2 - x} = \frac{A}{x} + \frac{B}{x-1} \quad \text{...(1)}$$

$$\frac{1}{x^2 - x} = \frac{A(x-1) + Bx}{x(x-1)} \left[\begin{array}{c}\text{by simplification} \\ \text{taking L.C.M. of R.H.S.}\end{array}\right]$$

Observe that, L.H.S. denominator and R.H.S. denominator are equal,

∴ L.H.S. numerator = R.H.S. numerator

$$1 = A(x-1) + Bx \quad \text{...(2)}$$

Put x = 0, in Equation (1)

[To find A from Equation (1), put denominator of A equal to Zero]

$$1 = A(0-1) + B(0)$$
$$1 = A(-1)$$
$$1 = -A \qquad \Rightarrow \mathbf{A = -1}$$

Put x – 1 = 0 ⇒ x = 1, in Equation (2)

[To find B from Equation (1), put denominator of B equal to Zero]

$$1 = A(0) + B(1)$$
$$1 = 0 + B$$
$$1 = B \qquad \Rightarrow \mathbf{B = 1}$$

Substitute these values (A = – 1, B = 1) in Equation (1)

∴ Equation (1) becomes,

$$\frac{1}{x^2 - x} = \frac{-1}{x} + \frac{1}{x-1}$$

$$\mathbf{\frac{1}{x^2 - x} = \frac{1}{x-1} - \frac{1}{x}} \quad ✓ \quad \left(\begin{array}{c}\text{Taking positive} \\ \text{terms as I}^{st}\end{array}\right)$$

This is required solution.

Ex. 4.1.2 (W-05, S-10, 2 Marks)

Resolve into partial fraction : $\frac{x-2}{x(x-1)}$ or $\frac{x-2}{x^2 - x}$

☑ **Soln. :** First find out all possible factors of denominator

Since, $x^2 - x = x(x-1)$

Consider, $$\frac{x-2}{x^2 - x} = \frac{x-2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \quad \text{...(1)}$$

$$\therefore \frac{x-2}{x^2 - x} = \frac{A(x-1) + Bx}{x(x-1)} \left[\begin{array}{c}\text{By simplification} \\ \text{taking L.C.M of R.H.S.}\end{array}\right]$$

Observe that, L.H.S. denominator and R.H.S. denominator are equal.

∴ L.H.S. numerator = R.H.S. numerator

$$x - 2 = A(x-1) + Bx \quad \text{...(2)}$$

Put x = 0, in Equation (2)

[To find A from Equation (1), put denominator of A equal to Zero]

$$0 - 2 = A(0-1) + B(0)$$
$$-2 = A(-1) + 0$$
$$-2 = -A \qquad \Rightarrow \mathbf{A = 2}$$

Put x – 1 = 0 ⇒ x = 1, in Equation (2)

[To find B from Equation (1), put denominator of B equal to Zero]

$$1 - 2 = A(0) + B(1)$$
$$-1 = 0 + B$$
$$-1 = B \qquad \Rightarrow \mathbf{B = -1}$$

Substitute these values (A = 2, B = – 1) in Equation (1).

∴ Equation (1) becomes,

$$\frac{x-2}{x^2 - x} = \frac{2}{x} + \frac{-1}{x-1}$$

$$\therefore \mathbf{\frac{x-2}{x^2 - x} = \frac{2}{x} - \frac{1}{x-1}} \quad ✓ \text{ This is required solution.}$$

Ex. 4.1.3 (W-07, W-08, S-09, S-17, 4 Marks)

Resolve into partial fraction : $\frac{x+4}{x^2 + x}$ OR $\frac{x+4}{x(x+1)}$

☑ **Soln. :** First find out all possible factors of denominator

We can write,

$$x^2 + x = x(x+1)$$

$$\therefore \frac{x+4}{x^2 + x} = \frac{x+4}{x(x-1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$\frac{x+4}{x^2 + x} = \frac{A}{x} + \frac{B}{x+1} \quad \text{...(1)}$$

$$\frac{x+4}{x^2 + x} = \frac{A(x+1) + Bx}{x(x+1)} \left[\begin{array}{c}\text{by simplification} \\ \text{taking L.C.M. of R.H.S.}\end{array}\right]$$

Observe that, L.H.S. denominator and R.H.S. denominator are equal.

∴ L.H.S. numerator = R.H.S. numerator

$$x + 4 = A(x+1) + Bx \quad \text{...(2)}$$

Put x = 0, in Equation (1),

[To find A from Equation (1), put denominator of A equal to Zero]

$$0 + 4 = A(0+1) + B(0)$$
$$4 = A(1) \quad \Rightarrow \quad \mathbf{A = 4}$$

Put x + 1 = 0 ⇒ x = – 1, in Equation (2),

[To find B from Equation (1), put denominator of B equal to Zero]

$$-1 + 4 = A(0) + B(-1)$$
$$3 = 0 - B$$
$$3 = -B \quad \Rightarrow \quad \mathbf{B = -3}$$

Substitute these values (A = 4, B = – 3) in Equation (1)

∴ Equation (1) becomes,

$$\therefore \frac{x+4}{x^2+x} = \frac{4}{x} + \frac{-3}{x+1}$$

$$\mathbf{\frac{x+4}{x^2+x} = \frac{4}{x} - \frac{3}{x+1}}$$ ✓ This is required solution.

Ex. 4.1.4 **W-15, 2 Marks.**

Resolve into the partial fraction $\frac{1}{x^3+3x^2+2x}$.

☑ **Soln. :** First find out all possible factors of denominator.

We know,

$$x^3 + 3x^2 + 2x = x(x^2 - 3x + 2) = x(x+1)(x+2)$$

Consider,

$$\frac{1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} \quad \ldots(1)$$

$$\frac{1}{x(x+1)(x+2)} = \frac{A(x+1)(x+2) + Bx(x+2) + Cx(x+1)}{x(x+1)(x+2)}$$

[by simplification taking L.C.M. of R.H.S.]

Observe that, L.H.S. denominator and R.H.S. denominator are equal.

∴ L.H.S. numerator = R.H.S. numerator

$$\therefore \quad 1 = A(x+1)(x+2) + B\,x(x+2) + C\,x(x+1) \quad \ldots(2)$$

Put x = 0, in Equation (2)

[To find A from Equation (1), put denominator of A equal to Zero]

$$1 = A(0+1)(0+2) + B(0)(0+2) + C(0)(0+1)$$
$$1 = A(1)(2) + 0 + 0$$
$$1 = 2A$$
$$\therefore A = \frac{1}{2} \quad \Rightarrow \quad A = \frac{1}{2}$$

Put x + 1 = 0 ⇒ x = – 1, in Equation (2)

[To find B from Equation (1), put denominator of B equal to Zero]

$$1 = A(0)(-1+2) + B(-1)(-1+2) + C(-1)(0)$$
$$= 0 + B(-1)(1) + 0$$
$$1 = -B \quad \Rightarrow \quad B = -1$$

Put x + 2 = 0 ⇒ x = – 2, in Equation (2)

[To find C from Equation (1), put denominator of C equal to Zero]

$$1 = A(-2+1)(0) + B(-2)(0) + C(-2)(-2+1)$$
$$= 0 + 0 + C(-2)(-1) = 2C$$
$$\therefore C = \frac{1}{2} \quad \Rightarrow \quad C = \frac{1}{2}$$

Substitute these values ($A = \frac{1}{2}$, B = – 1, $C = \frac{1}{2}$) in Equation (1).

∴ Equation (1) becomes,

$$\frac{1}{x(x+1)(x+2)} = \frac{\frac{1}{2}}{x} + \frac{-1}{x+1} + \frac{\frac{1}{2}}{x+2}$$

$$\therefore \mathbf{\frac{1}{x(x+1)(x+2)} = \frac{1}{2}\frac{1}{x} - \frac{1}{x+1} + \frac{1}{2}\frac{1}{x+2}}$$

Ex. 4.1.5 **(S-07, 4 Marks)**

Resolve into partial fraction : $\frac{8x-4}{3x^2-2x-1}$

☑ **Soln. :** First find out all possible factors of denominator

Since,

$$3x^2 - 2x - 1 = 3x^2 - 3x + x - 1$$

Note the factors : – 3 + 1 = – 2 and – 3 × 1 = – 3

$$= 3x(x-1) + (x-1)$$
$$3x^2 - 2x - 1 = (x-1)(3x+1)$$
$$\therefore \frac{8x-4}{3x^2-2x-1} = \frac{8x-4}{(x-1)(3x+1)}$$

Consider,
$$\frac{8x-4}{3x^2-2x-1} = \frac{A}{x-1} + \frac{B}{3x+1} \quad \ldots(1)$$

$$\frac{8x-4}{3x^2-2x-1} = \frac{A(3x+1) + B(x-1)}{(x-1)(3x+1)}$$

[by simplification taking L.C.M. of R.H.S.]

Observe that, L.H.S. denominator and R.H.S. denominator are equal.

∴ L.H.S. numerator = R.H.S. numerator

$$\therefore \quad 8x - 4 = A(3x+1) + B(x-1) \quad \ldots(2)$$

Put x – 1 = 0 ⇒ x = 1, in Equation (2),

[To find A from Equation (1), put denominator of A equal to Zero]

$$8(1) - 4 = A((3 \times 1) + 1) + B(0)$$
$$8 - 4 = A(3+1) + 0$$
$$4 = A(4)$$

$$4 = 4A$$

$$A = \frac{4}{4} \quad \Rightarrow \quad \mathbf{A = 1}$$

Put $3x + 1 = 0 \Rightarrow 3x = -1 \Rightarrow x = \frac{-1}{3}$, in Equation (2),

[To find B from Equation (1), put denominator of B equal to Zero]

$$\therefore\ 8\left(\frac{-1}{3}\right) - 4 = A(0) + B\left(\frac{-1}{3} - 1\right)$$

$$\frac{-8}{3} - 4 = 0 + B\left(\frac{-1-3}{3}\right)$$

$$\frac{-8-(4\times 3)}{3} = B\left(\frac{-4}{3}\right)$$

$$\frac{-8-12}{3} = B\left(\frac{-4}{3}\right)$$

$$\frac{-20}{3} = B\left(\frac{-4}{3}\right)$$

$$\therefore\ \frac{-4}{3}B = \frac{-20}{3}$$

$$B = \frac{-20}{3} \times \frac{3}{-4} \quad \Rightarrow \quad \mathbf{B = 5}$$

Substitute these values (A = 1, B = 5) in Equation (1),

$\therefore$ Equation (1) becomes,

$$\mathbf{\frac{8x-4}{3x^2-2x-1} = \frac{1}{x-1} + \frac{5}{3x+1}} \checkmark$$

This is required solution.

Ex. 4.1.6 (Q. 2(b), W-17, S-22, 4 Marks)

Resolve into partial fractions : $\frac{x+3}{(x-1)(x+1)(x+5)}$

☑ **Soln. :** Consider,

$$\frac{x+3}{(x-1)(x+1)(x+5)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+5} \quad \ldots(1)$$

$$\frac{x+3}{(x-1)(x+1)(x+5)} = \frac{A(x+1)(x+5) + B(x-1)(x+5) + C(x-1)(x+1)}{(x-1)(x+1)(x+5)}$$

(By simplification taking LCM of RHS)

Observe that, L.H.S. denominator and R.H.S. denominator are equal.

$\therefore$ L.H.S. numerator = R.H.S. numerator

$$x + 3 = A(x+1)(x+5) + B(x-1)(x+5) + C(x-1)(x+1) \ldots(2)$$

Put $x - 1 = 0 \Rightarrow x = 1$, in equation (2),

[To find A from equation (1), put denominator of A equal to zero]

$$\therefore\ 1 + 3 = A(1+1)(1+5) + B(0)(1+5) + C(0)(1+1)$$

$$4 = A(2)(6) + 0 + 0$$

$$4 = A(12)$$

$$4 = 12A$$

$$\frac{4}{12} = A$$

$$A = \frac{4}{12} = \frac{1}{3}$$

$$\Rightarrow A = \frac{1}{3}$$

Put $x + 1 = 0 \Rightarrow x = -1$, in equation (2),

[To find B, from equation (1), put denominator of B equal to zero]

$$-1 + 3 = A(0)(-1+5) + B(-1-1)(-1+5) + C(-1-1)(0)$$

$$2 = 0 + B(-2)(4) + 0$$

$$2 = B(-8)$$

$$\frac{2}{-8} = B$$

$$\therefore\ B = \frac{2}{-8}$$

$$\Rightarrow B = \frac{-1}{4}$$

Put $x + 5 = 0 \Rightarrow x = -5$ in equation (2),

[To find C, from equation (1) put denominator of C equal to zero]

$$-5 + 3 = A(-5+1)(0) + B(-5-1)(0) + C(-5-1)(-5+1)$$

$$-2 = 0 + 0 + C(-6)(-4)$$

$$-2 = C(24)$$

$$\frac{-2}{24} = C \qquad \therefore\ C = \frac{-2}{24}$$

$$\Rightarrow C = \frac{-1}{12}$$

Substitute these values $\left(A = \frac{1}{3}, B = \frac{-1}{4}, C = \frac{-1}{12}\right)$ in equation (1)

$$\therefore \frac{x+3}{(x-1)(x+1)(x+5)} = \frac{\frac{1}{3}}{x-1} + \frac{\frac{-1}{4}}{x+1} + \frac{\frac{-1}{12}}{x+5}$$

$$\mathbf{\frac{x+3}{(x-1)(x+1)(x+5)} = \frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{4}\cdot\frac{1}{x+1} - \frac{1}{12}\cdot\frac{1}{x+5}}$$

…Ans.

Ex. 4.1.7 (W-06, 4 Marks)

Resolve into partial fractions : $\frac{2x-1}{(x+2)(x^2-1)}$.

☑ **Soln. :** First find out all possible factors of denominator

Consider,

$$\frac{2x-1}{(x+2)(x^2-1)} = \frac{2x-1}{(x+2)(x-1)(x+1)} \qquad \because\ x^2 - 1 = (x)^2 - (1)^2 = (x-1)(x+1)$$

$$\therefore \frac{2x-1}{(x+2)(x^2-1)} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{x+1} \quad \ldots(1)$$

$$\therefore \frac{2x-1}{(x+2)(x^2-1)} = \frac{A(x-1)(x+1)+B(x+2)(x+1)+C(x+2)(x-1)}{(x+2)(x-2)(x+1)}$$

[by simplification taking L.C.M. of R.H.S.]

Observe that, L.H.S. denominator and R.H.S. denominator are equal.

$\therefore$ L.H.S. numerator = R.H.S. numerator

$$\therefore \quad 2x-1 = A(x-1)(x+1)+B(x+2)(x+1) + C(x+2)(x-1) \quad \ldots(2)$$

Put x + 2 = 0 ⇒ x = – 2, in Equation (2),

[To find A from Equation (1), put denominator of A equal to Zero]

$$\therefore \quad 2(-2)-1 = A(-2-1)(-2+1)+B(0)(-2+1) + C(0)(-2-1)$$

$$-4-1 = A(-3)(-1)+0+0$$

$$-5 = 3A \quad \Rightarrow \quad \mathbf{A = \frac{-5}{3}}$$

Put x – 1 = 0 ⇒ x = 1, in Equation (2)

[To find B from Equation (1), put denominator of B equal to Zero]

$$2(1)-1 = A(0)(1+1)+B(1+2)(1+1) + C(1+2)(0)$$

$$2-1 = 0+B(3)(2)+0$$

$$1 = B(6) \quad \Rightarrow \quad \mathbf{B = \frac{1}{6}}$$

Put x + 1 = 0 ⇒ x = – 1, in Equation (2)

[To find C from Equation (1), put denominator of C equal to Zero]

$$2(-1)-1 = A(-1-1)(0)+B(-1+2)(0) + C(-1+2)(-1-1)$$

$$-2-1 = 0+0+C(1)(-2)$$

$$-3 = C(-2)$$

$$-3 = -2C$$

$$C = \frac{-3}{-2} \quad \Rightarrow \quad \mathbf{C = \frac{3}{2}}$$

Substitute these values $\left(A = \frac{-5}{3}, B = \frac{1}{6}, C = \frac{3}{2}\right)$ in Equation (1),

$$\therefore \frac{2x-1}{(x+2)(x^2-1)} = \frac{\frac{-5}{3}}{x+2} + \frac{\frac{1}{6}}{x-1} + \frac{\frac{3}{2}}{x+1}$$

$$\therefore \quad \mathbf{\frac{2x-1}{(x+2)(x^2-1)} = \frac{-5}{3} \cdot \frac{1}{x+2} + \frac{1}{6} \cdot \frac{1}{x-1} + \frac{3}{2} \cdot \frac{1}{x+1}} \checkmark$$

This is required solution.

Ex. 4.1.8 (W-08, S-17, Q. 2(b), S-18, 4 Marks)

Resolve into partial fractions : $\frac{x^2+1}{x(x^2-1)}$.

☑ **Soln. :** First find out all possible factors of denominator

Since, $x^2 - 1 = (x)^2 - (1)^2$

$x^2 - 1 = (x-1)(x+1)$ [∵ $a^2 - b^2 = (a-b)(a+b)$]

$$\therefore \frac{x^2+1}{x(x^2-1)} = \frac{x^2+1}{x(x-1)(x+1)}$$

$$\therefore \frac{x^2+1}{x(x^2-1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \quad \ldots(1)$$

$$\therefore \frac{x^2+1}{x(x^2-1)} = \frac{A(x-1)(x+1)+Bx(x+1)+Cx(x-1)}{x(x-1)(x+1)}$$

[by simplification taking L.C.M. of R.H.S.]

Observe that, L.H.S. denominator and R.H.S. denominator are equal.

$\therefore$ L.H.S. numerator = R.H.S. numerator

$$x^2+1 = A(x-1)(x+1)+B\,x(x+1)+C\,x(x-1) \quad \ldots(2)$$

Put x = 0, in Equation (2),

[To find A from Equation (1), put denominator of A equal to Zero]

$$0+1 = A(0-1)(0+1)+B(0)(0+1)+C(0)(0-1)$$

$$1 = A(-1)(1)+0+0$$

$$1 = -A \quad \Rightarrow \quad \mathbf{A = -1}$$

Put x – 1 = 0 ⇒ x = 1, in Equation (2),

[To find B from Equation (1), put denominator of B equal to Zero]

$$(1)^2+1 = A(0)(1+1)+B(1)(1+1)+C(1)(0)$$

$$1+1 = 0+B(1)(2)+0$$

$$2 = 2B$$

$$B = \frac{\cancel{2}}{\cancel{2}} = 1 \quad \Rightarrow \quad \mathbf{B = 1}$$

Put x + 1 = 0 ⇒ x = – 1, in Equation (2),

[To find C from Equation (1), put denominator of C equal to Zero]

$$(-1)^2+1 = A(-1-1)(0)+B(-1)(0) + C(-1)(-1-1)$$

$$1+1 = 0+0+C(-1)(-2)$$

$$2 = C(2)$$

$$\therefore \quad C = \frac{\cancel{2}}{\cancel{2}} = 1 \quad \Rightarrow \quad \mathbf{C = 1}$$

Substitute these values (A = – 1, B = 1, C = 1) in Equation (1)

∴ Equation (1) becomes,

$$\frac{x^2+1}{x(x^2-1)} = \frac{-1}{x} + \frac{1}{x-1} + \frac{1}{x+1}$$

$$\therefore \quad \mathbf{\frac{x^2+1}{x(x^2-1)} = \frac{1}{x-1} + \frac{1}{x+1} - \frac{1}{x}} \checkmark$$

This is required solution.

Ex. 4.1.9 (W-12, 4 Marks)

Resolve into partial fractions : $\frac{x^2+1}{x(x^2-1)}$.

☑ **Soln. :** First find out all possible factors of denominator

Since, $x^2 - 1 = (x)^2 - (1)^2 = (x-1)(x+1)$

$$\frac{x^2+1}{x(x^2-1)} = \frac{x^2+1}{x(x-1)(x+1)}$$

Consider, $\frac{x^2+1}{x(x^2-1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$ …(1)

$$\therefore \frac{x^2+1}{x(x^2-1)} = \frac{A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

[by simplification taking L.C.M. of R.H.S.]

Observe that, L.H.S. denominator and R.H.S. denominator are equal.

∴ L.H.S. numerator = R.H.S. numerator

$$x^2 + 1 = A(x-1)(x+1) + B(x)(x+1) + C\,x(x-1) \quad \text{…(2)}$$

Put x = 0, in Equation (2)

[To find A from Equation (1), put denominator of A equal to Zero]

$$0 + 1 = A(0-1)(0+1) + B(0)(0+1) + C(0)(0-1)$$
$$1 = A(-1)(1) + 0 + 0$$
$$1 = -A \quad \Rightarrow \quad \mathbf{A = -1}$$

Put x – 1 = 0 ⇒ x = 1, in Equation (2)

[To find B from Equation (1), put denominator of B equal to Zero]

$$(1)^2 + 1 = A(0)(1+1) + B(1)(1+1) + C(1)(0)$$
$$1 + 1 = 0 + B(1)(2) + 0$$
$$2 = 2B$$
$$B = \frac{\cancel{2}}{\cancel{2}} = 1 \quad \Rightarrow \quad \mathbf{B = 1}$$

Put x + 1 = 0 ⇒ x = – 1, in Equation (2)

[To find C from Equation (1), put denominator of C equal to Zero]

$$(-1)^2 + 1 = A(-1-1)(0) + B(-1)(0) + C(-1)(-1-1)$$
$$1 + 1 = 0 + 0 + C(-1)(-2)$$
$$2 = 2C$$
$$\therefore \quad C = \frac{\cancel{2}}{\cancel{2}} = 1 \quad \Rightarrow \quad \mathbf{C = 1}$$

Substitute these values (A = – 1, B = 1, C = 1) in Equation (1)

∴ Equation (1) becomes,

$$\frac{x^2+1}{x(x^2-1)} = \frac{-1}{x} + \frac{1}{x-1} - \frac{1}{x+1}$$

$$\therefore \quad \mathbf{\frac{x^2+1}{x(x^2-1)} = \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x+1}} \checkmark \quad \text{(Consider positive term first)}$$

This is required solution.

Ex. 4.1.10 (W-12, 4 Marks)

Resolve into partial fraction : $\frac{\tan\theta + 1}{(\tan\theta + 2)(\tan\theta + 3)}$

☑ **Soln. : Given :** $\frac{\tan\theta + 1}{(\tan\theta + 2)(\tan\theta + 3)}$

Put tan θ = x, given term becomes,

$$\therefore \quad \mathbf{\frac{x+1}{(x+2)(x+3)}}$$

Now, consider, $\frac{x+1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$ …(1)

$$\frac{x+1}{(x+2)(x+3)} = \frac{A(x+3) + B(x+2)}{(x+2)(x+3)}$$

[by simplification taking L.C.M. of R.H.S.]

Observe that, L.H.S. denominator and R.H.S. denominator are equal.

∴ L.H.S. numerator = R.H.S. numerator

$$x + 1 = A(x+3) + B(x+2) \quad \text{…(2)}$$

Put x + 2 = 0 ⇒ x = – 2, in Equation (2),

[To find A from Equation (1), put denominator of A equal to Zero]

$$-2 + 1 = A(-2+3) + B(0)$$
$$-1 = A(1) \quad \Rightarrow \quad \mathbf{A = -1}$$

Put x + 3 = 0 ⇒ x = – 3, in Equation (2),

[To find B from Equation (1), put denominator of B equal to Zero]

$$-3 + 1 = A(0) + B(-3+2)$$
$$-2 = 0 + B(-1)$$
$$-2 = -B \quad \Rightarrow \quad \mathbf{B = 2}$$

Substitute these values (A = – 1, B = 2) in Equation (1)

Equation (1) becomes,

$$\therefore \frac{x+1}{(x+2)(x+3)} = \frac{-1}{x+2} + \frac{2}{x+3}$$

$$\frac{x+1}{(x+2)(x+3)} = \frac{2}{x+3} - \frac{1}{x+2} \quad \text{(Taking positive terms first)}$$

Since, $x = \tan\theta$

$$\therefore \quad \frac{\tan\theta + 1}{(\tan\theta + 2)(\tan\theta + 3)} = \frac{2}{\tan\theta + 3} - \frac{1}{\tan\theta + 2} \checkmark$$

Exercise 4.2

Ex. 4.2.1 : Resolve into partial fraction : $\frac{x^2}{(x+1)(x-2)^2}$

Soln. : First find out all possible factors of denominator

Consider,

$$\frac{x^2}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \quad \text{…(1)}$$

$$\frac{x^2}{(x+1)(x-2)^2} = \frac{A(x-2)^2 + B(x+1)(x-2) + C(x+1)}{(x+1)(x-2)^2}$$

[by simplification taking L.C.M. of R.H.S.]

Observe that, L.H.S. denominator and R.H.S. denominator are equal.

L.H.S. numerator = R.H.S. numerator

$$\therefore \quad x^2 = A(x-2)^2 + B(x+1)(x-2) + C(x+1) \quad \text{…(2)}$$

Put $x + 1 = 0 \Rightarrow x = -1$, in Equation (2),

[To find A from Equation (1), put denominator of A equal to Zero]

$$(-1)^2 = A(-1-2)^2 + B(0)(-1-2) + C(0)$$
$$1 = A(-3)^2 + 0 + 0$$
$$1 = A(9)$$
$$1 = 9A \quad \Rightarrow \quad A = \frac{1}{9}$$

Put $x - 2 = 0 \Rightarrow x = 2$, in Equation (2),

[To find C from Equation (1), put denominator of C equal to Zero]

$$(2)^2 = A(0) + B(2+1)(0) + C(2+1)$$
$$4 = 0 + 0 + C(3)$$
$$4 = 3C \quad \Rightarrow \quad C = \frac{4}{3}$$

Put $x = 0$, in Equation (2),

[To find B from Equation (1), put denominator of B equal to Zero]

$$(0)^2 = A(0-2)^2 + B(0+1)(0-2) + C(0+1)$$
$$0 = A(-2)^2 + B(1)(-2) + C(1)$$
$$0 = A(4) + B(-2) + C$$

Since, $A = \frac{1}{9}$ and $C = \frac{4}{3}$

$$\therefore \quad 0 = \frac{1}{9} \times 4 + B(-2) + \frac{4}{3}; \quad 0 = \frac{4}{9} - 2B + \frac{4}{3}$$

$$0 = \left(\frac{4}{9} + \frac{4}{3}\right) - 2B; \quad \therefore 2B = \frac{4}{9} + \frac{4}{3}$$

$$2B = \frac{4+12}{9}$$

$$2B = \frac{16}{9} \Rightarrow B = \frac{16}{9} \times \frac{1}{2} = \frac{8}{9} \quad \Rightarrow \quad B = \frac{8}{9}$$

Substitute these values $\left(A = \frac{1}{9}, B = \frac{8}{9}, C = \frac{4}{3}\right)$ in Equation (1).

$$\frac{x^2}{(x+1)(x-2)^2} = \frac{\frac{1}{9}}{x+1} + \frac{\frac{8}{9}}{x-2} + \frac{\frac{4}{3}}{(x-2)^2}$$

$$\frac{x^2}{(x+1)(x-2)^2} = \frac{1}{9} \cdot \frac{1}{x+1} + \frac{8}{9} \cdot \frac{1}{x-2} + \frac{4}{3} \cdot \frac{1}{(x-2)^2} \checkmark$$

This is required solution.

Exercise 4.3

Ex. 4.3.1 (S-09, 4 Marks)

Resolve into partial fraction : $\frac{2x-1}{(x-1)(x^2+1)}$.

Soln. : Given : $\frac{2x-1}{(x-1)(x^2+1)}$

Here, $x^2 + 1$ can not be factorize further.

$$\text{Consider,} \quad \frac{2x-1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \quad \text{…(1)}$$

$$\frac{2x-1}{(x-1)(x^2+1)} = \frac{A(x^2+1) + (Bx+C)(x-1)}{(x-1)(x^2+1)}$$

[by simplification taking L.C.M. of R.H.S.]

Observe that, L.H.S. denominator and R.H.S. denominator are equal.

$\therefore$ L.H.S. numerator = R.H.S. numerator

$$\therefore \quad 2x - 1 = A(x^2+1) + (Bx + C)(x-1) \quad \text{…(2)}$$

Put $x - 1 = 0 \Rightarrow x = 1$, in Equation (2),

[To find A from Equation (1), put denominator of A equal to Zero]

$$2(1) - 1 = A[(1)^2 + 1] + [B(1) + C](0)$$
$$2 - 1 = A(1+1) + 0$$
$$1 = A(2)$$
$$\therefore \quad 1 = 2A \quad \Rightarrow \quad A = \frac{1}{2}$$

Put $x = 0$, in Equation (2),

$$2(0) - 1 = A[(0)^2 + 1) + [B(0) + C](0-1)$$
$$0 - 1 = A(1) + C(-1)$$
$$-1 = A - C$$
$$-1 = \frac{1}{2} - C \quad \left[\because A = \frac{1}{2}\right]$$
$$\therefore \quad C = \frac{1}{2} + 1 = \frac{3}{2} \quad \Rightarrow \quad C = \frac{3}{2}$$

Put x = – 1, in Equation (2),

$$2(-1)-1 = A[(-1)^2+1]+[B(-1)+C](-1-1)$$

$$-2-1 = A(1+1)+(-B+C)(-2)$$

$$-3 = A(2)+2B-2C$$

$$-3 = \left(\frac{1}{2}\times 2\right)+2B-\left(2\times\frac{3}{2}\right) \quad \left[\because\ A=\frac{1}{2},\ C=\frac{3}{2}\right]$$

$$-3 = 1+2B-3$$

$$\therefore \cancel{-3}-1+\cancel{3} = 2B$$

$$-1 = 2B \Rightarrow \mathbf{B = \frac{-1}{2}}$$

Substitute these values $\left(A=\frac{1}{2},\ B=\frac{-1}{2},\ C=\frac{3}{2}\right)$ in Equation (1)

Equation (1) becomes,

$$\frac{2x-1}{(x-1)(x^2+1)} = \frac{\frac{1}{2}}{x-1}+\frac{\frac{-1}{2}x+\frac{3}{2}}{x^2+1}$$

$$\mathbf{\frac{2x-1}{(x-1)(x^2+1)} = \frac{1}{2}\left[\frac{1}{x-1}+\frac{-x+3}{x^2+1}\right]} \ ✓$$

This is required solution.

Ex. 4.3.2 **W-10, S-16, 4 Marks.**

Resolve into partial fractions : $\frac{2x+1}{x^2(x+1)}$

☑ **Soln. : Given :** $\frac{2x+1}{x^2(x+1)}$

Consider,

$$\frac{2x+1}{x^2(x+1)} = \frac{A}{x+1}+\frac{Bx+C}{x^2} \quad \text{...(1)}$$

$$\frac{2x+1}{x^2(x+1)} = \frac{Ax^2+(Bx+C)(x+1)}{x^2(x+1)}$$

[by simplification taking L.C.M. of R.H.S.]

Observe that L.H.S. denominator and R. H. S. denominator are equal

∴ L.H.S. numerator = R.H.S. numerator

$$\therefore 2x+1 = Ax^2+(Bx+C)(x+1) \quad \text{...(2)}$$

Put x + 1 = 0 ⇒ x = – 1, in Equation (2),

[To find A from Equation (1), put denominator of A equal to Zero]

$$2(-1)+1 = A(-1)^2+[B(-1)+C](0)$$

$$-2+1 = A(1)+0$$

$$-1 = A \Rightarrow \mathbf{A = -1}$$

Put x = 0, in Equation (2),

$$\therefore 2(0)+1 = A(0)+[B(0)+C](0+1)$$

$$0+1 = 0+C(1)$$

$$1 = C \Rightarrow \mathbf{C = 1}$$

Put x = 1, in Equation (2),

$$2(1)+1 = A(1)^2+[B(1)+C](1+1)$$

$$2+1 = A+(B+C)(2)$$

$$3 = A+2B+2C$$

$$3 = -1+2B+2(1) \quad [\because\ A=-1,\ C=1]$$

$$3 = -1+2B+2$$

$$3+1-2 = 2B$$

$$2 = 2B \Rightarrow B = \frac{2}{2} \Rightarrow \mathbf{B = 1}$$

Substitute these values (A = –1, B = 1, C = 1) in Equation (1)

Equation (1) becomes,

$$\therefore \frac{2x+1}{x^2(x+1)} = \frac{-1}{x+1}+\frac{x+1}{x^2}$$

$$\therefore \mathbf{\frac{2x+1}{x^2(x+1)} = \frac{x+1}{x^2}-\frac{1}{x+1}} \ ✓ \quad \text{...(3)}$$

Note : If we use, $\frac{2x+1}{x^2(x+1)} = \frac{A}{x+1}+\frac{B}{x}+\frac{C}{x^2}$

Then from Equation (3) observe that

$$\frac{2x+1}{x^2(x+1)} = \frac{x}{x^2}+\frac{1}{x^2}-\frac{1}{x+1} = \frac{-1}{x+1}+\frac{1}{x}+\frac{1}{x^2}$$

Means we get, A = –1, B = 1, C = 1

Ex. 4.3.3 **.W-06, 4 Marks.**

Resolve into partial Fractions : $\frac{2x-3}{(x+1)(x^2+4)}$

☑ **Soln. : Given :** $\frac{2x-3}{(x+1)(x^2+4)}$

Here x^2+4 cannot factorize further,

$$\text{Consider, } \frac{2x-3}{(x+1)(x^2+4)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+4} \quad \text{...(1)}$$

$$\frac{2x-3}{(x+1)(x^2+4)} = \frac{A(x^2+4)+(Bx+C)(x+1)}{(x+1)(x^2+4)}$$

[by simplification taking L.C.M. of R.H.S.]

Observe that, L.H.S. denominator and R.H.S. denominator are equal.

∴ L.H.S. numerator = R.H.S. numerator

$$\therefore 2x-3 = A(x^2+4)+(Bx+C)(x+1) \quad \text{...(2)}$$

Put x = – 1, in Equation (2),

[To find A from Equation (1), put denominator of A equal to Zero]

$$\therefore 2(-1)-3 = A[(-1)^2+4]+[B(-1)+C](-1+1)$$

$$-2-3 = A(1+4)+(-B+C)(0)$$
$$-5 = A(5)+0$$
$$-5 = 5A \Rightarrow A = \frac{-5}{5} = -1 \qquad \Rightarrow \mathbf{A = -1}$$

Put, x = 0, in Equation (2),

$$2(0)-3 = A(0+4)+[B(0)+C](0+1)$$
$$0-3 = A(4)+C(1)$$
$$-3 = 4A+C \qquad (\therefore A=-1)$$
$$-3 = 4(-1)+C \Rightarrow C = -3+4 = 1 \Rightarrow \mathbf{C = 1}$$

Put, x = 1, in Equation (2)

$$2(1)-3 = A[(1)^2+4]+[B(1)+C](1+1)$$
$$2-3 = A(1+4)+(B+C)(2)$$
$$-1 = A(5)+2B+2C$$
$$-1 = 5A+2B+2C$$
$$-1 = 5(-1)+2B+2(1) \qquad [\because A=-1, C=1]$$
$$-1 = -5+2B+2$$
$$\therefore -1+5-2 = 2B$$
$$2 = 2B \Rightarrow B = \frac{2}{2} = 1 \qquad \Rightarrow \mathbf{B = 1}$$

Substitute these values (A = – 1, B = 1, C = 1) in Equation (1),

Equations (1) becomes,

$$\frac{2x-3}{(x+1)(x^2+4)} = \frac{-1}{x+1}+\frac{x+1}{x^2+4}$$

Taking positive term first

$$\mathbf{\frac{2x-3}{(x+1)(x^2+4)} = \frac{x+1}{x^2+4} - \frac{1}{x+1}} \checkmark$$

This is required solution.

Ex. 4.3.4 (S-15, 4 Marks)

Resolve into partial fractions : $\frac{x^2+36x+6}{(x-1)(x^2+2)}$

☑ **Soln. :** Given, $\frac{x^2+36x+6}{(x-1)(x^2+2)}$

Here x^2+2 cannot factorize further

Consider,

$$\frac{x^2+36x+6}{(x-1)(x^2+2)} = \frac{A}{x-1}+\frac{Bx+C}{(x^2+2)} \qquad \text{...(1)}$$

$$\frac{x^2+36x+6}{(x-1)(x^2+2)} = \frac{A(x^2+2)+(Bx+C)(x-1)}{(x-1)(x^2+2)}$$

[by simplification taking L.C.M. of R.H.S.]

Observe that, L.H.S. denominator and R.H.S. denominator are equal.

∴ L.H.S. numerator = R.H.S. numerator

$$\therefore x^2+36x+6 = A(x^2+2)+(Bx+C)(x-1)$$

Put x – 1 = 0 ⇒ x = 1, in Equation (2)

[To find A from Equation (1), put denominator of A equal to Zero]

$$\therefore (1)^2+36(1)+6 = A((1)^2+2)+(B(1)+C)(0)$$
$$1+36+6 = A(1+2)+0$$
$$43 = 3A \qquad \left[\because \mathbf{A = \frac{43}{3}}\right] \Rightarrow A = \frac{43}{3}$$

Put x = 0, in Equation (2)

$$\therefore (0)^2+36(0)+6 = A(0+2)+(B(0)+C)(0-1)$$
$$6 = 2A+C(-1)$$
$$6 = 2\left(\frac{43}{3}\right)-C \qquad \left[\because A = \frac{43}{3}\right]$$
$$6 = \frac{86}{3}-C$$
$$\therefore C = \frac{86}{3}-6 = \frac{86-18}{3}$$
$$C = \frac{83}{3} \qquad \Rightarrow \mathbf{C = \frac{68}{3}}$$

Put x = – 1, in Equation (2)

$$\therefore (-1)^2+36(-1)+6 = A((-1)^2+2)+(B(-1)+C)(-1-1)$$
$$1-36+6 = A(1+2)+(-B+C)(-2)$$
$$-29 = 3A+2B-2C$$
$$-29 = 3\left(\frac{43}{3}\right)+2B-2\left(\frac{68}{3}\right) \qquad \left[\because A = \frac{43}{3}, C = \frac{68}{3}\right]$$
$$-29 = 43+2B-\frac{136}{3}$$
$$2B = -29-43+\frac{136}{3}$$
$$2B = \frac{-29(3)-43(3)+136}{3}$$
$$2B = \frac{-80}{3} \Rightarrow B = \frac{-80}{3}\times\frac{1}{2} \qquad \Rightarrow \mathbf{B = \frac{-40}{3}}$$

Substitute these values $\left(A = \frac{43}{3}, B = \frac{-40}{3}, C = \frac{68}{3}\right)$ in Equation (1).

∴ Equation (1) becomes,

$$\frac{x^2+36x+6}{(x-1)(x^2+2)} = \frac{\frac{43}{3}}{x-1}+\frac{\frac{-40}{3}x+\frac{68}{3}}{x^2+2}$$

$$\mathbf{\frac{x^2+36x+6}{(x-1)(x^2+2)} = \frac{1}{3}\left[\frac{43}{x-1}-\frac{40x-68}{x^2+2}\right]} \checkmark$$

This is required solution.

Ex. 4.3.5 (W-12, 4 Marks)

Resolve into partial fraction : $\frac{1}{x^3 - 1}$

☑ **Soln. : Given :** $\frac{1}{x^3 - 1}$

First find out all possible factors of denominator

Note that, $(x^3 - 1) = (x - 1)(x^2 + x + 1)$

$[\because a^3 - b^3 = (a - b)(a^2 + ab + b^2)]$

$\therefore \frac{1}{x^3 - 1} = \frac{1}{(x - 1)(x^2 + x + 1)}$ and

$x^2 + x + 1$, can not factories further.

Consider,

$$\frac{1}{x^3 - 1} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1} \quad \text{...(1)}$$

$$\frac{1}{x^3 - 1} = \frac{A(x^2 + x + 1) + (Bx + C)(x - 1)}{(x - 1)(x^2 + x + 1)}$$

Observe that, L.H.S. denominator and R.H.S. denominator are equal.

$\therefore$ L.H.S. numerator = R.H.S. numerator

$$\therefore \; 1 = A(x^2 + x + 1) + (Bx + C)(x - 1) \quad \text{...(2)}$$

Put x – 1 = 0 ⇒ x = 1, in Equation (2)

[To find A from Equation (1), put denominator of A equal to Zero]

$$1 = A[(1)^2 + 1 + 1] + [B(1) + C](0)$$
$$1 = A(1 + 1 + 1) + 0$$
$$1 = A(3) \Rightarrow A = 3 \qquad \Rightarrow \mathbf{A = \frac{1}{3}}$$

Put x = 0, in Equation (2)

$$1 = A[(0)^2 + (0) + 1] + [B(0) + C] \times (0 - 1)$$
$$1 = A(1) + C(-1)$$
$$1 = \frac{1}{3}(1) - C \qquad \left(\because A = \frac{1}{3}\right)$$
$$1 - \frac{1}{3} = -C$$
$$\therefore \frac{2}{3} = -C \qquad \Rightarrow \mathbf{C = \frac{-2}{3}}$$

Put x = – 1, in Equation (2)

$$1 = A\left[(-1)^2 - \cancel{1} + \cancel{1}\right] + [B(-1) + C](-1 - 1)$$
$$1 = A(1) + (-B + C)(-2)$$
$$1 = \left(\frac{1}{3}\right)(1) + \left(-B - \frac{2}{3}\right)(-2) \qquad \left(\because A = \frac{1}{3}, C = \frac{-2}{3}\right)$$
$$1 = \frac{1}{3} + 2B + \frac{4}{3}$$
$$\therefore 1 - \frac{1}{3} - \frac{4}{3} = 2B$$
$$\frac{3 - 1 - 4}{3} = 2B \Rightarrow \frac{-2}{3} = 2B$$
$$\Rightarrow B = \frac{-2}{3} \times \frac{1}{2} \Rightarrow \mathbf{B = \frac{-1}{3}}$$

Substitute these values $\left(A = \frac{1}{3}, B = \frac{-1}{3}, C = \frac{-2}{3}\right)$ in Equation (1),

Equations (1) becomes,

$$\frac{1}{x^3 - 1} = \frac{\frac{1}{3}}{x - 1} + \frac{\left(\frac{-1}{3}\right)x + \left(\frac{-2}{3}\right)}{x^2 + x + 1}$$

$$\therefore \mathbf{\frac{1}{x^3 - 1} = \frac{1}{3}\left[\frac{1}{x - 1} - \frac{x + 2}{x^3 + x + 1}\right]} \checkmark$$

This is required solution.

Ex. 4.3.6 (W-15, 4 Marks)

Resolve into partial fractions $\frac{x}{x^3 + 1}$

☑ **Soln. : Given :** $\frac{x}{x^3 + 1}$

First find out all possible factors of denominator

Note that,

$x^3 + 1 = (x + 1)(x^2 - x + 1)$ and

$[\because a^3 + b^3 = (a + b)(a^2 - ab + b^2)]$

$x^2 - x + 1$ can't factories further.

Consider,

$$\frac{x}{x^3 + 1} = \frac{A}{x + 1} + \frac{Bx + C}{(x^2 - x + 1)} \quad \text{...(1)}$$

$$\frac{x}{x^3 + 1} = \frac{A(x^2 - x + 1) + (Bx + C)(x + 1)}{(x + 1)(x^2 - x + 1)}$$

[by simplification taking L.C.M. of R.H.S]

Observe that, L.H.S. denominator and R.H.S. denominator are equal.

$\therefore$ L.H.S. numerator = R.H.S. numerator

$$\therefore \; x = A(x^2 - x + 1) + (Bx + C)(x + 1) \quad \text{...(2)}$$

Put x + 1 = 0 ⇒ x = – 1, in Equation (2)

[To find A from Equation (1), put denominator of A equal to Zero]

$$-1 = A[(-1)^2 - (-1) + 1] + [B(-1) + C] \times (0)$$
$$-1 = A(1 + 1 + 1) + 0$$
$$-1 = A(3) \Rightarrow -1 = 3A \quad \Rightarrow \mathbf{A = \frac{-1}{3}}$$

Put x = 0, in Equation (2)

$$0 = A[(0)^2 - (0) + 1] + [B(0) + C] \times (0 + 1)$$
$$0 = A(1) + C(1)$$
$$0 = A + C$$

$$0 = \frac{-1}{3} + C \qquad \left(\because A = -\frac{1}{3}\right)$$

$$\therefore C = \frac{1}{3} \qquad \Rightarrow \mathbf{C = \frac{1}{3}}$$

Put x = 1, in Equation (2)

$$1 = A\left[(1)^2 - (\not{1}) + \not{1}\right] + [B(1) + C](1 + 1)$$

$$= A(1) + (B + C)(2)$$

$$1 = A + 2B + 2C$$

$$1 = \frac{-1}{3} + 2B + 2\left(\frac{1}{3}\right) \qquad \left(\because A = -\frac{1}{3}, C = \frac{1}{3}\right)$$

$$1 = \frac{-1}{3} + 2B + \frac{2}{3}$$

$$\therefore 1 + \frac{1}{3} - \frac{2}{3} = 2B \Rightarrow \frac{3 + 1 - 2}{3} = 2B$$

$$\frac{2}{3} = 2B \Rightarrow B = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3} \qquad \Rightarrow \mathbf{B = \frac{1}{3}}$$

Substitute these values $\left(A = -\frac{1}{3}, B = \frac{1}{3}, C = \frac{1}{3}\right)$ in Equation (1),

$$\frac{x}{x^3 + 1} = \frac{\frac{-1}{3}}{x + 1} + \frac{\frac{1}{3}x + \frac{1}{3}}{x^2 - x + 1}$$

$$\mathbf{\frac{x}{x^3 + 1} = \frac{1}{3}\left[\frac{-1}{x + 1} + \frac{x + 1}{x^2 - x + 1}\right]}$$

This is required solution.

Ex. 4.3.7 **S-17, 4 Marks)**

Resolve into the partial fractions :

$$\frac{x^2 + 1}{(x + 1)(x^2 + 4)}$$

☑ **Soln. : Given :** $\frac{x^2 + 1}{(x + 1)(x^2 + 4)}$

Here $x^2 + 4$ cannot factorize further,

Consider,

$$\frac{x^2 + 1}{(x + 1)(x^2 + 4)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 4} \qquad \text{…(1)}$$

$$\frac{x^2 + 1}{(x + 1)(x^2 + 4)} = \frac{A(x^2 + 4) + (Bx + C)(x + 1)}{(x + 1)(x^2 + 4)}$$

(by Simplification taking L.C.M. on R.H.S)

Observe that L.H.S. denominator and R.H.S. denominator are equal.

∴ L.H.S. numerator = R.H.S. numerator

$$x^2 + 1 = A(x^2 + 4) + (Bx + C)(x + 1) \qquad \text{…(2)}$$

Put x + 1 = 0 ⇒ x = – 1 in Equation (2),

[To find A from Equation (1), put denominator of A equal to zero]

$$\therefore (-1)^2 + 1 = A((-1)^2 + 4) + (B(-1) + C)(0)$$

$$1 + 1 = A(1 + 4) + 0$$

$$2 = 5A \qquad \therefore \mathbf{A = \frac{2}{5}}$$

Put x = 0 in Equation (2),

$$0 + 1 = A(0 + 4) + (B(0) + C)(0 + 1)$$

$$1 = 4A + C(1)$$

$$1 = 4\left(\frac{2}{5}\right) + C \qquad \left(\because A = \frac{2}{5}\right)$$

$$\therefore 1 = \frac{8}{5} + C$$

$$1 - \frac{8}{5} = C; \qquad \frac{5 - 8}{5} = C \qquad \Rightarrow \frac{-3}{5} = C$$

$$\therefore C = \frac{-3}{5}$$

Put x = 1, in Equation (2)

$$(1)^2 + 1 = A((1)^2 + 4) + (B(1) + C)(1 + 1)$$

$$1 + 1 = A(1 + 4) + (B + C)(2)$$

$$2 = 5A + 2B + 2C$$

$$2 = 5\left(\frac{2}{5}\right) + 2B + 2\left(\frac{-3}{5}\right)$$

$$2 = 2 + 2B - \frac{6}{5}$$

$$2 - 2 + \frac{6}{5} = 2B; \qquad \frac{6}{5} = 2B$$

$$\frac{6}{5 \times 2} = B \qquad \therefore B = \frac{3}{5} \quad A = \frac{2}{5}, B = \frac{3}{5}$$

Substitute these value $\left(A = \frac{2}{5} B = \frac{3}{5} \text{ and } C = \frac{-3}{5}\right)$

In Equation (1)

$$\frac{x^2 + 1}{(x + 1)(x^2 + 4)} = \frac{\frac{2}{5}}{x + 1} + \frac{\frac{3}{5}x - \frac{3}{5}}{x^2 + 4}$$

$$\mathbf{\frac{x^2 + 1}{(x + 1)(x^2 + 4)} = \frac{1}{5}\left[\frac{2}{x+1} + \frac{3x - 3}{x^2 + 4}\right]} \checkmark \qquad \text{…Ans.}$$

Exercise 4.4

Ex. 4.4.1 **.S-15, 4 Marks.**

Resolve into partial fractions $\frac{x^3 + 1}{x^2 + 6x}$

☑ **Soln. :** Given, $\frac{x^3 + 1}{x^2 + 6x}$

Observe that, degree of numerator > degree of denominator $\frac{x^3 + 1}{x^2 + 6x}$ is a improper fraction.

By actual division : Divided $(x^3 + 1)$ by $x^2 + 6x$ and this improper fraction convert into proper fraction.

$$\text{Divisor } x^2+6x \overline{)\, x^3+1\,} \quad \text{Quotient: } x-6$$

$$\begin{array}{l} x^3 \pm 6x^2 \\ \hline -6x^2 + 1 \\ \hline \mp 6x^2 \mp 36x \\ \hline 36x+1 \leftarrow \text{Reminder} \end{array}$$

$$\therefore x^3+1 = (\text{Divisor} \times \text{Quotient}) + \text{Reminder}$$

$$x^3+1 = (x^2+bx)(x-6)+(36x+1)$$

$$\frac{x^3+1}{x^2+6x} = \frac{(x^2+6x)(x-6)(36x+1)}{x^2+6x}$$

$$= \frac{(x^2+6x)(x-6)}{x^2+6x} + \frac{36x+1}{x^2+6x}$$

$$\frac{x^3+1}{x^2+6x} = (x-6)+\frac{36x+1}{x^2+6x} \quad \text{...(1)}$$

Consider,

$$\frac{36x+1}{x^2+6x} = \frac{36x+1}{x(x+6)} = \frac{A}{x}+\frac{B}{x+6} \quad \text{...(2)}$$

$$\frac{36x+1}{x^2+6x} = \frac{A(x+6)+Bx}{x(x+6)} \quad \left[\text{By simplification taking L.C.M. of R.H.S}\right]$$

Observe that L.H.S. denominator and R.H.S. denominator are equal.

∴ L.H.S numerator = R.H.S. numerator

$$\therefore 36x+1 = A(x+6)+(Bx) \quad \text{...(3)}$$

Put $x = 0$, in Equation (3)

[To find A from Equation (2), put denominator of A equal to Zero]

$$\therefore 36(0)+1 = A(0+6)+B(0)$$

$$0+1 = A(6)+0$$

$$1 = 6A \qquad \Rightarrow \mathbf{A = \frac{1}{6}}$$

Put $x+6=0 \Rightarrow x=-6$, in Equation (3)

$$36(-6)+1 = A(0)+B(-6)$$

$$-216+1 = 0-6B$$

$$-215 = -6B$$

$$\frac{-215}{-6} = B \qquad \Rightarrow \mathbf{B = \frac{215}{6}}$$

Substitute these valves $\left(A=\frac{1}{6},\ B=\frac{215}{6}\right)$ in Equation (2)

$$\frac{36x+1}{x^2+6x} = \frac{\frac{1}{6}}{x}+\frac{\frac{215}{6}}{x+6}$$

$$\frac{36x+1}{x^2+6x} = \frac{1}{6}\left[\frac{1}{x}+\frac{215}{x+6}\right] \quad \left[\text{Common out } \left(\frac{1}{6}\right) \text{ from R.H.S.}\right]$$

Substitute this in Equation (1),

$$\therefore \mathbf{\frac{x^3+1}{x^2+6x} = (x-6)+\frac{1}{6}\left[\frac{1}{x}+\frac{215}{x+6}\right]} \ ✓$$

This is required Solution.

Ex. 4.4.2 (W-09, 4 Marks)

Resolve into partial fraction : $\frac{x^3+x}{x^2-9}$

☑ **Soln. : Given :** $\frac{x^3+x}{x^2-9}$

Observe that, degree of Numerator > degree of Denominator.

$\therefore \frac{x^3+x}{x^2-9}$ is a improper fraction.

By a actual division : divide (x^3+x) by (x^2-9) and this improper fraction convert into proper fraction.

$$\text{Divisor} \longrightarrow x^2-9 \overline{)\, x^3+x\,} \quad \text{Quotient: } x$$

$$\begin{array}{l} -x^3 - 9x \\ \quad - \quad + \\ \hline 10x \leftarrow \text{Reminder} \end{array}$$

$$x^3+x = (\text{Divisor} \times \text{Quotient}) + \text{Reminder}$$

$$x^3+x = (x^2-9)(x)+10x$$

$$\therefore \frac{x^3+x}{x^2-9} = \frac{(x^2-9)(x)+10x}{x^2-9}$$

$$\frac{x^3+x}{x^2-9} = \frac{(x^2-9)(x)}{x^2-9}+\frac{10x}{x^2-9}$$

$$\frac{x^3+x}{x^2-9} = x+\frac{10x}{x^2-9} \quad \text{...(1)}$$

Now, since

$$x^2-9 = (x-3)(x+3)[\because a^2-b^2=(a-b)(a+b)]$$

Consider,

$$\frac{10x}{x^2-9} = \frac{A}{x-3}+\frac{B}{x+3} \quad \text{...(2)}$$

$$\frac{10x}{x^2-9} = \frac{A(x+3)+B(x-3)}{(x-3)(x+3)}$$

[By simplification taking L.C.M of R.H.S]

Observe that, L.H.S. denominator and R.H.S. denominator are equal.

$\therefore$ L.H.S. numerator = R.H.S. numerator

$$\therefore \quad 10x = A(x+3) + B(x-3) \quad \text{...(3)}$$

Put $x - 3 = 0 \Rightarrow x = 3$, in Equation (3),

[To find A from Equation (2), put denominator of A equal to Zero]

$$\therefore \quad 10(3) = A(3+3) + B(0)$$

$$30 = A(6) + 0$$

$$30 = 6A \Rightarrow A = \frac{30}{6} \quad \Rightarrow \quad \mathbf{A = 5}$$

Put $x + 3 = 0 \Rightarrow x = -3$, in Equation (3),

$$10(-3) = A(0) + B(-3-3)$$

$$-30 = 0 + B(-6)$$

$$-30 = -6B \Rightarrow B = \frac{-30}{-6} \quad \Rightarrow \quad \mathbf{B = 5}$$

Substitute these values (A = 5, B = 5) in Equation (2),

$$\frac{10x}{x^2-9} = \frac{5}{x-3} + \frac{5}{x+3}$$

Substitute this in Equation (1),

$$\therefore \quad \frac{x^3+x}{x^2-9} = x + 5\left[\frac{1}{x-3} + \frac{1}{x+3}\right] \checkmark$$

This is required solution.

Chapter Ends…

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Chapter 5 : TRIGONOMETRIC RATIOS
(Compound, Allied, Multiple and Submultiple Angle)

Exercise 5.1

Ex. 5.1.1 : Find value of cos 15° or cos $\left(\frac{\pi}{12}\right)^c$

☑ **Soln. :** Since 45° – 30° = 15°

$\therefore \quad \cos 15° = \cos(\underbrace{45°}_{A} - \underbrace{30°}_{B})$ …(1)

➤ **Use formula for eqn. (1)**

…[cos (A – B) = cos A · cos B + sin A · sin B]

$$\cos(45° - 30°) = \underbrace{\cos(45°)}_{\frac{1}{\sqrt{2}}} \cdot \underbrace{\cos(30°)}_{\frac{\sqrt{3}}{2}} + \underbrace{\sin(45°)}_{\frac{1}{\sqrt{2}}} \cdot \underbrace{\sin(30°)}_{\frac{1}{2}}$$ …(2)

➤ **Use standard values for eqn. (2)**

$$\ldots\left[\cos 45° = \frac{1}{\sqrt{2}}, \cos 30° = \frac{\sqrt{3}}{2}, \sin 45° = \frac{1}{\sqrt{2}}, \sin 30° = \frac{1}{2}\right]$$

$$\cos(15°) = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$\left(\because \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}\right)$$

$$\mathbf{\cos(15°) = \frac{\sqrt{3}+1}{2\sqrt{2}}}\ ✓$$ …Ans.

Ex. 5.1.2 (S-2017, 4 Marks, Q. 1(c), W-22, 2 Marks)

Find value of cos 75° or cos $\left(\frac{5\pi}{12}\right)^c$

☑ **Soln. :** Since 35° + 45° = 75°

$\therefore \quad \cos 75° = \cos(\underbrace{30°}_{A} + \underbrace{45°}_{B}) = 75°$ …(1)

➤ **Use formula for eqn. (1)**

…[cos (A + B) = cos A · cos B – sin A · sin B]

$$\cos(30° + 45°) = \underbrace{\cos(30°)}_{\frac{\sqrt{3}}{2}} \cdot \underbrace{\cos(45°)}_{\frac{1}{\sqrt{2}}} - \underbrace{\sin(30°)}_{\frac{1}{2}} \cdot \underbrace{\sin(45°)}_{\frac{1}{\sqrt{2}}}$$ …(2)

➤ **Use standard values for eqn. (2)**

$$\ldots\left[\cos 30° = \frac{\sqrt{3}}{2}, \cos 45° = \frac{1}{\sqrt{2}}, \sin 30° = \frac{1}{2}, \sin 45° = \frac{1}{\sqrt{2}}\right]$$

$$\cos(75°) = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$\left(\because \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}\right)$$

$$\mathbf{\cos(15°) = \frac{\sqrt{3}-1}{2\sqrt{2}}}\ ✓$$ …Ans.

Ex. 5.1.3 (W-15, 2 Marks)

Without using calculator find the value of cos (3660).

☑ **Soln. :**

$$\cos(3660°) = \cos[(40 \times 90°) + 60°]$$

$$= \cos(60°) = \frac{1}{2}$$

[40 × 90° → even × 90° no change in cos and (40 × 90°) + 60° it is in 1st quadrant]

$$\therefore \quad \mathbf{\cos(3660°) = \frac{1}{2}}\ ✓$$

Ex. 5.1.4 (W-15, 4 Marks)

Find value of $\dfrac{\sec^2 135°}{\cos(-240°) - 2\sin(930°)}$

☑ **Soln. :**

$\sec(135°) = \sec(90° + 45°) = -\operatorname{cosec}(45°)$ …(1)

➤ **Use standard value for eqn. (1) :**

… [cosec (45°) = $\sqrt{2}$]

$\sec(135°) = -\sqrt{2}$ …(2)

$\therefore \sec^2(135°) = (-\sqrt{2})^2 = 2$

$\cos(-240°) = \cos(240°)$

$= \cos(270° - 30°) = -\sin(30°)$ …(3)

➤ **Use standard value for eqn. (3) : …[sin (30°) = $\frac{1}{2}$]**

$\cos(-240°) = -\frac{1}{2}$ …(4)

$\sin(930°) = \sin(10 \times 90° + 30°) = \sin(30°)$ …(5)

➤ **Use standard value for eqn. (5) :… [sin (30°) = $\frac{1}{2}$]**

$\mathbf{\sec(930°) = \frac{1}{2}}$ …(6)

From Equation (2), (4) and (6),

$$\frac{\sec^2(135°)}{\cos(-240) - 2\sin(930°)} = \frac{2}{-\frac{1}{2} - 2\left(\frac{1}{2}\right)}$$

$$= \frac{2}{-\frac{1}{2}-1} = \frac{2}{-\frac{3}{2}} = 2 \times \left(\frac{2}{-3}\right) = \frac{-4}{3}$$

$$\frac{\sec^2 (135°)}{\cos (-240) - 2 \sin (930°)} = \frac{-4}{3} \checkmark \quad \text{...Ans.}$$

Ex. 5.1.5 : Prove that : $\sin^2 \theta \cdot \sec^2 \theta + \sin^2 \theta \cdot \text{cosec}^2 \theta = \sec^2 \theta$

☑ **Soln. :** L.H.S. $= \sin^2 \theta \cdot \sec^2 \theta + \sin^2 \theta \cdot \text{cosec}^2 \theta$ **...(1)**

➤ **Use for eqn. (1) :** ... $\left[\sec \theta = \frac{1}{\cos \theta} \text{ and } \text{cosec } \theta = \frac{1}{\sin \theta}\right]$

$$= \sin^2 \theta \cdot \frac{1}{\cos^2 \theta} + \sin^2 \theta \cdot \frac{1}{\sin^2 \theta}$$

$$= \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} = \tan^2 \theta + 1 = 1 + \tan^2 \theta \quad \text{...(2)}$$

➤ **Use standard formula for eqn. (2) :**

... $[1 + \tan^2 \theta = \sec^2 \theta]$

L.H.S $= \sec^2 \theta$

L.H.S. = R.H.S. ✓ **...Hence proved**

Ex. 5.1.6 : Prove that : $\cos \left(\frac{\pi}{2} - \theta\right) = \sin \theta$

☑ **Soln. :**

$$\text{L.H.S} = \cos \left(\underbrace{\frac{\pi}{2}}_{A} - \underbrace{\theta}_{B}\right) \quad \text{...(1)}$$

➤ **Use formula for eqn. (1)**

...[cos (A – B) = cos A · cos B + sin A · sin B]

$$\cos\left(\frac{\pi}{2} - \theta\right) = \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} \cdot \cos \theta + \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} \cdot \sin \theta \quad \text{...(2)}$$

➤ **Use standard values for eqn. (2)**

$$\ldots\left[\cos\left(\frac{\pi}{2}\right) = 0 \text{ and } \sin\left(\frac{\pi}{2}\right) = 1\right]$$

$$= [0 \times \cos \theta] + [1 \times \sin \theta] = 0 + \sin \theta = \sin \theta$$

$\therefore$ **cos** $\left(\frac{\pi}{2} - \theta\right) =$ **sin θ** ✓ **...Hence proved.**

Ex. 5.1.7 (W-2012, 2 Marks)

Prove that : cos (π + θ) = – cos θ

☑ **Soln. :**

$$\text{L.H.S.} = \cos (\underbrace{\pi}_{A} + \underbrace{\theta}_{B}) \quad \text{...(1)}$$

➤ **Use formulae for eqn. (1)**

... [cos (A + B) = cos A · cos B – sin A · sin B]

$$\cos (\pi + \theta) = \underbrace{\cos \pi}_{-1} \cdot \cos \theta - \underbrace{\sin \pi}_{0} \cdot \sin \theta \quad \text{...(2)}$$

➤ **Use standard values for eqn. (2)**

... [cos π = – 1 and sin π = 0]

$$= [(-1) \times \cos \theta] - [0 \times \sin \theta]$$

$$\cos (\pi + \theta) = -\cos \theta - 0$$

$\therefore$ **cos (π + θ) = – cos θ** ✓ **...Hence proved.**

Ex. 5.1.8 : Prove that : tan (π + θ) = tan θ

☑ **Soln. :**

$$\text{Since, } \tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\therefore \tan (\pi + \theta) = \frac{\sin (\pi + \theta)}{\cos (\pi + \theta)} \quad \text{...(1)}$$

➤ **Use transformation formulae for eqn. (1)**

... [sin (π + θ) = – sin θ, cos (π + θ) = – cos θ]

$$= \frac{-\sin \theta}{-\cos \theta} = \tan \theta$$

$\therefore$ **tan (π + θ) = tan θ** ✓ **...Hence proved.**

Ex. 5.1.9 : Prove that : sec (π + θ) = – sec θ

☑ **Soln. :**

$$\text{Since, } \sec \theta = \frac{1}{\cos \theta}$$

$$\therefore \sec (\pi + \theta) = \frac{1}{\cos (\pi + \theta)} \quad \text{...(1)}$$

➤ **Use transformation formula for eqn. (1)**

... [cos (π + θ) = – cos θ]

$$= \frac{1}{-\cos \theta} = -\sec \theta$$

$\therefore$ **sec (π + θ) = – sec θ** ✓ **...Hence proved.**

Ex. 5.1.10 (W-16, 2 Marks)

Without using calculator find the value of sin (– 765°)

☑ **Soln.**

We know trigonometric function for (n × 90°) is :

(i) If n is even function no change in function

(ii) If n is odd then function changes to co-function.

$$\text{Since } \sin (-\theta) = -\sin \theta$$

$$\therefore \sin (-760°) = -\sin (765°)$$

$$= -\sin [(9 \times 90°) - 45°] \quad \text{...(1)}$$

➤ **Use for eqn. (1)**

$\left[\begin{array}{c} 9 \times 90^\circ \rightarrow \text{odd} \times 90^\circ, \text{ sin changes to cos and } (9 \times 90^\circ) - 45^\circ, \\ \text{it is in } 1^{st} \text{ quadrant. } \therefore \text{ It is positive.} \end{array}\right]$

$\therefore \sin(-760^\circ) = -\cos(45^\circ)$ …(2)

➤ **Use standard value for eqn. (2)** … $\left[\cos(45^\circ) = \frac{1}{\sqrt{2}}\right]$

$= \frac{1}{\sqrt{2}}$

$\therefore \sin(-760^\circ) = \frac{1}{\sqrt{2}}$ ✓ …Ans.

Ex. 5.1.11 **(S-15, 4 Marks)**

Without using calculator find the value of sin (150°) – tan (315°) + cos (300°) + $\sec^2$ (360°)

☑ **Soln. :**

We know trigonometric function for $(n \times 90^\circ)$ is,

(a) If n is even no change in function

(b) n is odd function changes to co-function.

To decide sign find the quadrant.

$\sin(150^\circ) = \sin(90^\circ + 60^\circ)$ …(1)

➤ **Use transformation formula for eqn. (1)**

… $\left[\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta\right]$

$= \cos 60^\circ$ …(2)

➤ **Use standard value in eqn. (2) :** … $\left[\cos 60^\circ = \frac{1}{2}\right]$

$\therefore \sin(150^\circ) = \frac{1}{2}$ …(3)

(ii) $\tan(315^\circ) = \tan[(3 \times 90^\circ) + 45^\circ]$

$\left[\begin{array}{c} 3 \times 90^\circ \rightarrow \text{odd} \times 90^\circ \\ \text{tan changes to cot and} \\ (3 \times 90^\circ) + 45^\circ \text{ it is in} \\ IV^{th} \text{ quadrant} \\ \text{tan negative} \end{array}\right]$

$= -\cot 45^\circ$ …(4)

➤ **Use standard value for Eqn. (4) :** … **[cot 45° = 1]**

$\therefore \tan(315^\circ) = -1$ …(5)

(iii) $\cos(300^\circ) = \cos[(3 \times 90^\circ) + 30^\circ]$

$\left[\begin{array}{c} 3 \times 90^\circ \rightarrow \text{odd} \times 90^\circ \\ \text{cos changes to sin and} \\ (3 \times 90^\circ) + 30^\circ \text{ it is in} \\ IV^{th} \text{ quadrant} \\ \text{cos positive} \end{array}\right]$

$= \sin 30^\circ$ …(6)

➤ **Use standard value for Eqn. (6) :** … $\left[\sin 30^\circ = \frac{1}{2}\right]$

$\therefore \cos(300^\circ) = \frac{1}{2}$ …(7)

(iv) $\sec^2(360^\circ) = \sec^2[(4 \times 90^\circ) + 0^\circ]$

$\left[\begin{array}{c} 4 \times 90^\circ \rightarrow \text{even} \times 90^\circ \\ \text{no changes in sec and} \\ (4 \times 90^\circ) + \theta \text{ it is in} \\ IV^{th} \text{ quadrant} \\ \text{sec positive} \end{array}\right]$

$= \sec^2(0^\circ)$ …(8)

➤ **Using standard value for Eqn. (8) :** … **[sec 0 = 1]**

Using all these values, Equations (3), (5), (7) and (9),

$\sec^2(360^\circ) = (1)^2 = 1$ …(9)

$\sin(150^\circ) - \tan(315^\circ) + \cos(300^\circ) + \sec^2(360^\circ)$

$= \frac{1}{2} - (-1) + \frac{1}{2} + 1 = \frac{1}{2} + 1 + \frac{1}{2} + 1 = 3$

$\mathbf{\sin(150^\circ) - \tan(315^\circ) + \cos(300^\circ) + \sec^2(360^\circ) = 3}$ ✓ …Ans.

Ex. 5.1.12 **W-16, 4 Marks)**

Without using calculator, find the value of :

tan (585°) . cot (– 495°) – cot (405°) · tan (– 495°)

☑ **Soln. :** We know trigonometric function for $(n \times 90^\circ)$ is,

(i) If n is even, then no change in function Or

(ii) If n is odd, then function changes to co-function

To decide sign, find the quadrant.

(I) $\tan(585^\circ) = \tan[(6 \times 90^\circ) + 45^\circ]$ …(1)

$\left[\begin{array}{c} 6 \times 90^\circ \rightarrow \text{even} \times 90^\circ, \text{ no change in tan and } ((6 \times 90^\circ) + 45^\circ) \\ \text{it is in third quadrant, } \therefore \text{ tan positive} \end{array}\right]$

$\therefore \tan(585^\circ) = \tan(45^\circ)$

➤ **Use formula in eqn. (1) :** … **(tan 45° = 1)**

$\therefore \tan(585^\circ) = 1$ …(2)

(II) Since $\cot(-\theta) = -\cot\theta$

$\therefore \cot(-495^\circ) = -\cot(-495^\circ)$

$= -\cot[(5 \times 90^\circ) + 45^\circ]$

$\left[\begin{array}{c} (5 \times 90^\circ) \rightarrow \text{odd} \times 90^\circ, \text{ cot changes to tan and} \\ 5 \times 90^\circ + 45 \\ \text{is in } II^{nd} \text{ quadrant so cot is negative} \end{array}\right]$

$\therefore \cot(-495^\circ) = [-\tan(45^\circ)] = \tan(45^\circ)$

$\cot(-495^\circ) = 1$ $(\because \tan 45^\circ = 1)$ …(3)

(III) $\cot(-405^\circ) = \cot[(4 \times 90^\circ) + 45^\circ]$

$\left[\begin{array}{c} 4 \times 90^\circ \rightarrow \text{even} \times 90^\circ, \text{ no changes in cot and} \\ (4 \times 90^\circ) + 45^\circ, \text{ is in } I^{st} \text{ quadrant so cot is positive} \end{array}\right]$

$= \cot(45^\circ)$

$\cot(-405^\circ) = 1$ $[\because \text{standard value } \cot 45^\circ = 1]$ …(4)

(IV) Since $\tan(-\theta) = -\tan\theta$

$\tan(-495^\circ) = -\tan(495^\circ)$

$= \; -\tan[(5\times 90°) + 45°]$

$\left[\begin{array}{l}(5\times 90°) \to \text{odd} \times 90°\text{, tan changes to cot and} \\ 5\times 90° + 45\text{, it is in II}^{nd}\text{ quadrant } \therefore \text{tan negative}\end{array}\right]$

$= \; -[-\tan 45°]$

$= \; \tan 45° \quad \left(\begin{array}{l}\because \text{ standard value} \\ \tan 45° = 1\end{array}\right)$

$\tan(-495°) \; = \; 1$ …(5)

Using values from Equation (2), (3) (4) and (5),

$\tan(585°)\cot(-495°) - \cot(-405°)\tan(-495°)$

$= \; (1)(1) - (1)(1)$

$= \; 1 - 1 = \mathbf{0}$ ✓ …Ans.

Ex. 5.1.13 : Show that $\cos 510° \cdot \cos 330° + \sin 390° \cdot \cos 120° = -1$

☑ **Soln. :**

$\cos(510°) \; = \; \cos[(6\times 90°) - 30°]$ [Note this step]

$= \; \cos(30°) = -\dfrac{\sqrt{3}}{2}$

$\left\{\begin{array}{l}6\times 90° \to \text{even} \times 90°\text{ no change} \\ \text{in cos and } (6\times 90° - 30°) \\ \text{is in II}^{nd}\text{ quadrant}\end{array}\right\}$

$\cos(330°) \; = \; \cos[(4\times 90°) - 30°]$

$\left\{\begin{array}{l}4\times 90° \to \text{even}\times 90°\text{ no change} \\ \text{in cos and it is in IV}^{th}\text{ quadrant}\end{array}\right\}$

$= \; \cos 30° = \dfrac{\sqrt{3}}{2}$

$\sin(390)° \; = \; \sin[(4\times 90°) + 30°]$

$\left\{\begin{array}{l}4\times 90° \to \text{even}\times 90°\text{ no change} \\ \text{in sin and } [(4\times 90°) + 30°] \\ \text{is in I}^{st}\text{ quadrant}\end{array}\right\}$

$= \; \sin 30° = \dfrac{1}{2}$

$\cos(120°) \; = \; \cos(90° + 30°)$ [It is in IInd quadrant]

$= \; -\sin 30° = -\dfrac{1}{2}$

Using these values,

$\cos 510° \cdot \cos 330° + \sin 390° \cdot \cos 120°$

$$= \left(-\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right) = \left(\frac{-3}{4}\right) + \left(\frac{-1}{4}\right)$$

$$= \frac{-3-1}{4} = \frac{-4}{4} = -1$$

$\therefore$ **cos 510° cos 330° + sin 390° · cos 120° = – 1** ✓

…Hence proved

Ex. 5.1.14 **S-10, 4 Marks**

Prove that,

$\cos A \cdot \cos(60° - A)\cos(60° + A) = \dfrac{1}{4}\cos 3A$

☑ **Soln. :**

L.H.S $= \cos A \cdot \cos(60° - A) \cdot \cos(60° + A)$ …(1)

➢ **Use formulae for eqn. (1)**

$\left[\begin{array}{l}\cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B \\ \ldots \text{ and } \cos(A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B\end{array}\right]$

$$= \cos A\,[\underbrace{\cos 60°}_{\frac{1}{2}} \cdot \cos A + \underbrace{\sin 60°}_{\frac{\sqrt{3}}{2}} \cdot \sin A]$$

$$\cdot\,[\underbrace{\cos 60°}_{\frac{1}{2}} \cdot \cos A - \underbrace{\sin 60°}_{\frac{\sqrt{3}}{2}} \cdot \sin A] \quad \ldots(2)$$

➢ **Use standard values for eqn. (2)**

$\ldots\left[\mathbf{\cos 60° = \dfrac{1}{2} \text{ and } \sin 60° = \dfrac{\sqrt{3}}{2}}\right]$

$$= \cos A\left[\underbrace{\frac{1}{2}\cdot\cos A}_{a} + \underbrace{\frac{\sqrt{3}}{2}\cdot\sin A}_{b}\right]\cdot\left[\underbrace{\frac{1}{2}\cdot\cos A}_{a} - \underbrace{\frac{\sqrt{3}}{2}\cdot\sin A}_{b}\right]$$

…(3)

➢ **Use for eqn. (3) :** … $\mathbf{[(a + b)\cdot(a - b) = a^2 - b^2]}$

$$= \cos A\left[\left(\frac{1}{2}\cos A\right)^2 - \left(\frac{\sqrt{3}}{2}\sin A\right)^2\right]$$

$$= \cos A\left[\frac{1}{4}\cos^2 A - \frac{3}{4}\sin^2 A\right]$$

$$= \frac{1}{4}\cos^3 A - \frac{3}{4}\cos A \cdot \sin^2 A \quad \ldots(4)$$

To prove convert R.H.S. into cos θ

➢ **Use formulae for eqn. (4)**

…[$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \sin^2\theta = 1 - \cos^2\theta$]

$$= \frac{1}{4}\cos^3 A - \frac{3}{4}\cos A\,(1 - \cos^2 A)$$

$$= \frac{1}{4}\cos^3 A - \frac{3}{4}\cos A + \frac{3}{4}\cos^3 A$$

$$= \left(\frac{1}{4}\cos^3 A + \frac{3}{4}\cos^3 A\right) - \frac{3}{4}\cos A = \cos^3 A - \frac{3}{4}\cos A$$

$$= \frac{1}{4}\cdot 4\cos^3 A - \frac{3}{4}\cos A \quad \text{(Note the adjustment)}$$

$$= \frac{1}{4}(4\cos^3 A - 3\cos A) \quad \ldots(5)$$

➢ **Use formula for eqn. (5):…[4cos^{3}θ – 3 cos θ = cos 3θ]**

$$\text{L.H.S.} = \frac{1}{4}\cos 3A$$

L.H.S. = R.H.S.

$\therefore$ **cos A · cos (60° – A) cos (60° + A) = $\frac{1}{4}$ cos 3 A** ✓

…Hence Proved

Ex. 5.1.15 : Show that : cos (A + B) · cos (A – B) = $\cos^2 A - \sin^2 B$

☑ **Soln. :**

L.H.S= cos (A + B) · cos (A – B) …(1)

➢ **Use formulae for eqn. (1)**

… [cos (A + B) = cos A· cos B – sin A· sin B and cos (A – B) = cos A· cos B + sin A· sin B]

$$= [\underbrace{\cos A\cdot \cos B}_{a} - \underbrace{\sin A\cdot \sin B}_{b}]\,[\underbrace{\cos A\cdot \cos B}_{a} + \underbrace{\sin A \sin B}_{b}]$$

…(2)

➢ **Use for eqn. (2) :** **… [(a – b) (a + b) = $a^2 - b^2$]**

$= (\cos A \cdot \cos B)^2 - (\sin A \cdot \sin B)^2$

$= \cos^2 A \cdot \cos^2 B - \sin^2 A \cdot \sin^2 B$ …(3)

To prove obtain R.H.S. cos A and sin B only

➢ **Use for eqn. (3)**

$[\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \sin^2\theta,\ \sin^2\theta = 1 - \cos^2\theta]$

$= \cos^2 A\,[1 - \sin^2 B] - [1 - \cos^2 A]\sin^2 B$

$= \cos^2 A - \cancel{\cos^2 A \cdot \sin^2 B} - \sin^2 B + \cancel{\cos^2 A \cdot \sin^2 B}$

$= \cos^2 A - \sin^2 B$

L.H.S = R.H.S. ✓ **…Hence proved**

Ex. 5.1.16 **S-08, 2 Marks**

Evaluate : $\dfrac{\tan 66° + \tan 69°}{1 - \tan 66° \tan 69°}$

☑ **Soln. :** Taking, $\tan(\underbrace{66°}_{A} + \underbrace{69°}_{B})$ …(1)

➢ **Use formula for eqn. (1)**

$$\ldots\left[\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}\right]$$

Using this formula with A = 66° and B = 69°

$$\tan (66° + 69°) = \frac{\tan 66° + \tan 69°}{1 - \tan 66° \tan 69°}$$

$$\tan (135°) = \frac{\tan 66° + \tan 69°}{1 - \tan 66° \tan 69°}$$

i.e. $\dfrac{\tan 66° + \tan 69°}{1 - \tan 66° + \tan 69°} = \tan (135°)$

$= \tan (90° + 45°)$ [Note (90° + 45°) is in IInd quadrant] …(2)

➢ **Use transformation formula for eqn. (2)**

… [tan (90° + θ) = – cot θ]

= – cot (45°) …(3)

➢ **Use for eqn. (3) standard value : [cot 45° = 1]**

= – 1

$$\therefore\quad \frac{\tan 66° + \tan 69°}{1 - \tan 66° \tan 69°} = -1$$ ✓ **…Ans.**

Ex. 5.1.17 **(W-16, 4 Marks)**

If tan (x + y) = $\frac{1}{2}$ and tan (x – y) = $\frac{1}{3}$ find

(i) tan 2x (ii) tan 2y

☑ **Soln. :** **Given :**

$\tan(x + y) = \frac{1}{2}$ and $\tan (x - y) = \frac{1}{3}$ …(1)

➢ **(i) Use formula in eqn. (1)**

$$\ldots\left[\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}\right]$$

$$\therefore \tan [\underbrace{(x + y)}_{A} + \underbrace{(x - y)}_{B}] = \frac{\tan (x + y) + \tan (x - y)}{1 - \tan (x + y) \tan (x - y)}$$

➢ **Use values from eqn. (1)**

$$\tan [x + \cancel{y} + x - \cancel{y}] = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}} = \frac{\frac{3+2}{6}}{1 - \frac{1}{6}}$$

$$\tan (2x) = \frac{\frac{5}{6}}{\frac{6-1}{6}} = \frac{\frac{5}{6}}{5/6} = 1$$

tax 2x = 1 ✓ **…Ans.**

➢ **(ii) Use formula in eqn. (1)**

$$\ldots\left[\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A\ \tan B}\right]$$

$$\tan [\underbrace{(x + y)}_{A} - \underbrace{(x - y)}_{B}] = \frac{\tan(x + y) - \tan(x - y)}{1 + \tan (x + y)\tan (x - y)}$$

$$\tan [x + \cancel{y} - x + y] = \frac{\frac{1}{3} - \frac{1}{4}}{1 + \frac{1}{3}\cdot\frac{1}{4}}$$

$$\tan (2y) = \frac{\frac{4-3}{12}}{1 + \frac{1}{12}} = \frac{\frac{1}{12}}{\frac{12+1}{12}} = \frac{\frac{1}{\cancel{12}}}{\frac{13}{\cancel{12}}}$$

tan (2y) = $\frac{1}{13}$ ✓ **…Ans.**

Ex. 5.1.18 **.W-12, 2 Marks.**

Without using calculator find the value of tan 75° OR tan $\left(\frac{5\pi}{12}\right)$

☑ **Soln. :** $\tan(75°) = \tan(\underbrace{45°}_{A} + \underbrace{30°}_{B})$ …(1)

➢ **Use formula for eqn. (1)**

… $\left[\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}\right]$

Using this formula with A = 45° and B = 30°

$\therefore \quad \tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \cdot \tan 30°}$ …(2)

➢ **Use standard values for eqn. (2)**

… $\left[\tan 45° = 1, \ \tan 30° = \frac{1}{\sqrt{3}}\right]$

$$\tan(75°) = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \left(1 \times \frac{1}{\sqrt{3}}\right)}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}+1}{\cancel{\sqrt{3}}}}{\frac{\sqrt{3}-1}{\cancel{\sqrt{3}}}} \quad \text{[by simplification]}$$

$\therefore$ **tan (75°)** $= \frac{\sqrt{3}+1}{\sqrt{3}-1}$ ✓ …Ans.

Ex. 5.1.19 : If $\sin A = \frac{4}{5}$, $\cos A = \frac{3}{5}$, $\sin B = \frac{12}{13}$, and cos B = $\frac{5}{13}$, find the exact values of sin (A + B), cos (A + B), and tan (A + B)

☑ **Soln. :**

Given : $\sin A = \frac{4}{5}$; $\cos A = \frac{3}{5}$

$\sin B = \frac{12}{13}$; $\cos B = \frac{5}{13}$ …(1)

(i) Calculate : sin (A + B) …(2)

➢ **Use formula for eqn. (2)**

… [sin (A + B) = sin A cos B + cos A sin B]

sin (A + B) = sin A cos B + cos A sin B

$= \frac{4}{5} \cdot \frac{5}{13} + \frac{3}{5} \cdot \frac{12}{13}$ [Values form Equation (1)]

$= \frac{20}{65} + \frac{36}{65} = \frac{20+36}{65}$

$\Rightarrow$ **sin (A + B)** $= \frac{56}{65}$ ✓ …Ans.

(ii) Calculate : cos (A + B) …(3)

➢ **Use formula for eqn. (3)**

… [cos (A + B) = cos A cos B – sin A sin B]

cos (A + B) = cos A cos B – sin A sin B

$= \frac{3}{5} \cdot \frac{5}{13} - \frac{4}{5} \cdot \frac{12}{13}$ [values from Equation (1)]

$= \frac{15}{65} - \frac{48}{65} = \frac{15-48}{65}$

$\Rightarrow$ **cos (A + B) =** $-\frac{33}{65}$ ✓ …Ans.

(iii) $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$

$= \frac{\left(\frac{56}{\cancel{65}}\right)}{\left(-\frac{33}{\cancel{65}}\right)}$ (Results from (i) and (ii))

$\Rightarrow$ **tan (A + B) =** $-\frac{56}{33}$ ✓ …Ans.

Ex. 5.1.20

.S-2008, S-2009, S-2012, 4 Marks, Q. 3(a), S-19, 2 Marks.

tan 70° – tan 50° – tan 20° = tan70° tan 50° tan 20°

☑ **Soln. :**

▶ **Step I :** Taking, $\tan 70° = \tan(\underbrace{50°}_{A} + \underbrace{20°}_{B})$ …(1)

➢ **Use formula for eqn. (1)**

… $\left[\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}\right]$

$\therefore \quad \tan(50° + 20°) = \frac{\tan 50° + \tan 20°}{1 - \tan 50° \cdot \tan 20°}$

$\tan(70°) = \frac{\tan 50° + \tan 20°}{1 - \tan 50° \cdot \tan 20°}$

$\therefore$ tan 70° · [1 – tan 50° · tan 20°] = tan 50° + tan 20°

i.e. tan 50° + tan 20° = tan 70° [1 – tan 50° · tan 20°] …(2)

▶ **Step II :**

Now, To prove consider,

L.H.S. = tan 70° – tan 50° – tan 20°

tan 70° – tan 50° – tan 20° = tan 70° – [tan 50° + tan 20°]

= tan 70° – tan 70° [1 – tan 50° · tan 20°]

[From Equation (2)]

= $\cancel{\tan 70°} - \cancel{\tan 70°}$ + tan 70° · tan 50° · tan 20° = R.H.S.

L.H.S. = R.H.S.

$\therefore$ **tan 70° – tan 50° – tan 20° = tan 70° tan 50° tan 20°** ✓

…Hence proved.

Ex. 5.1.21 **W-16, 4 Marks.**

If $\tan A = \frac{1}{3}$, $\tan B = \frac{1}{4}$, where $0 < A < \frac{\pi}{2}$, $\pi < B < \frac{3\pi}{2}$

Find sin (A + B).

☑ **Soln. :** We know,

$$\sin (A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B \quad ...(1)$$

∴ First find sin A, cos B, cos A, sin B

Given $\tan A = \frac{1}{3}$, $\tan B = \frac{1}{4}$

▶ **Step I :** Since $\tan A = \frac{1}{3}; 0 < A < \frac{\pi}{2}$

$$\tan A = \frac{1}{3} = \frac{\text{Opposite side}}{\text{Adjacent side}}$$

By right angle triangle find third side of triangle as shown in Fig. P. 5.12.36

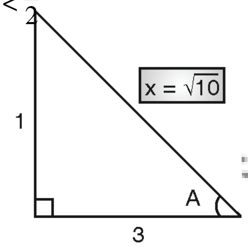

Fig. P. 5.12.36

By pythagoras triplet 1, 3, $\sqrt{10}$

OR

Suppose third side (hypotenuse) is x in right angle triangle

$$(1)^2 + (3)^2 = x^2$$

$$1 + 9 = x^2 \qquad \therefore x^2 = 10 \quad \therefore x = \sqrt{10}$$

Since $0 < A < \frac{\pi}{2}$, it is in first quadrant so both sin A, cos A are positive

$$\therefore \sin A = \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{\mathbf{1}}{\sqrt{\mathbf{10}}} \quad ...(2)$$

$$\cos A = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{\mathbf{3}}{\sqrt{\mathbf{10}}} \quad ...(3)$$

▶ **Step II :** Since $\tan B = \frac{1}{4}$, $\pi < B < \frac{3\pi}{2}$

$$\tan B = \frac{1}{4} = \frac{\text{Opposite side}}{\text{Adjacent side}}$$

By write angle triangle find third side of triangle as shown in Fig. P. 5.12.36(a)

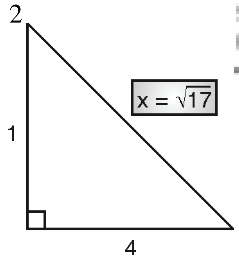

Fig. P. 5.12.36(a)

By pythagoras triplet 1, 4, $\sqrt{17}$

OR suppose third side (hypotenuse) is y in right angle triangle

$$(1)^2 + (3)^2 = y^2$$

$$1 + 16 = y^2 \qquad \therefore y^2 = 17$$

$$\therefore y = \sqrt{17}$$

since $\pi < B < \frac{3\pi}{2}$, it is in third quadrant, so both sin B, cos B are negative.

$$\therefore \sin B = \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{\mathbf{1}}{\sqrt{\mathbf{17}}} \quad ...(4)$$

$$\cos B = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{\mathbf{4}}{\sqrt{\mathbf{17}}} \quad ...(5)$$

▶ Step III : Substitute values from (2), (3), (4) and (5) in Equation (1)

$$\therefore \sin (A + B) = \frac{1}{\sqrt{10}} \cdot \frac{4}{\sqrt{17}} + \frac{3}{\sqrt{10}} \cdot \frac{1}{\sqrt{17}}$$

$$= \frac{4}{\sqrt{10} \cdot \sqrt{17}} + \frac{3}{\sqrt{10} \cdot \sqrt{17}} = \frac{4 + 3}{\sqrt{10} \cdot \sqrt{17}}$$

$$= \frac{7}{\sqrt{10}\sqrt{17}} = \frac{7}{\sqrt{10 \times 17}} = \frac{7}{\sqrt{170}}$$

$$\therefore \mathbf{\sin (A + B)} = \frac{\mathbf{7}}{\sqrt{\mathbf{170}}} \checkmark \quad \text{...Ans.}$$

Ex. 5.1.22 **(S-13, 4 Marks)**

Prove that for any angle θ

$$\sin 2\theta = 2 \sin \theta \cdot \cos \theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

☑ **Soln. :**

$$\sin 2\theta = \sin (\underbrace{\theta}_{A} + \underbrace{\theta}_{B}) \quad ...(1)$$

➤ **Use formula for eqn. (1)**

… [sin (A + B) = sin A· cos B + cos A· sin B]

$$= \sin \theta \cdot \cos \theta + \cos \theta \cdot \sin \theta$$

$$= \sin \theta \cdot \cos \theta + \sin \theta \cdot \cos \theta$$

$$\therefore \mathbf{\sin 2\theta = 2 \sin \theta \cdot \cos \theta} \checkmark \quad ...(2)$$

Now, $\sin 2\theta = 2 \sin \theta \cdot \cos \theta$

Multiply and divide R.H.S. by cos θ.

$$\therefore \sin 2\theta = 2 \sin \theta \cdot \cos \theta \times \frac{\cos \theta}{\cos \theta}$$

$$= \frac{2 \sin \theta \cdot \cos^2 \theta}{\cos \theta} \quad ...(3)$$

➤ **Use for eqn. (3) :** $\quad ...\left[\mathbf{\cos \theta = \frac{1}{\sec \theta}}\right]$

$$= 2 \frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\sec^2 \theta} \quad ...(4)$$

➤ **Use for eqn. (4) :** $\quad ...\left[\mathbf{\frac{\sin \theta}{\cos \theta} = \tan \theta}\right]$

$$= \frac{2 \tan \theta}{\sec^2 \theta} \quad ...(5)$$

➤ **For formula eqn. (5) :** **… [$\sec^2 \theta = 1 + \tan^2 \theta$]**

$$\mathbf{\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}} \quad ...(6)$$

from Equations (2) and (6)

$\therefore \sin 2\theta = 2 \sin\theta \cdot \cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$ ✓ ...Ans.

Ex. 5.1.23 **(W-08, S-09, 2 Marks)**
If $A = 30°$ verify that $\sin 3A = 3\sin A - 4\sin^3 A$.

☑ **Soln. :**

Given : $A = 30°$

$\sin 3A = \sin(3 \times 30°) = \sin 90°$ ...(1)

➢ **Use standard value for eqn. (1) :** ... ($\sin 90° = 1$)

$= 1 \quad \therefore \sin 3A = 1$ ...(2)

Now, R.H.S. $= 3\sin A - 4\sin^3 A$

$= 3\sin 30° - 4\sin^3 30°$

$= 3\sin 30° - 4(\sin 30°)^3$ ...(3)

➢ **Use standard value for eqn. (3) :** ...$\left(\sin 30° = \dfrac{1}{2}\right)$

$= 3\left(\dfrac{1}{2}\right) - 4\left(\dfrac{1}{2}\right)^3$

$= \dfrac{3}{2} - 4\left(\dfrac{1}{8}\right) = \dfrac{3}{2} - \dfrac{1}{2} = \dfrac{2}{2}$

$3\sin A - 4\sin^3 A = 1$ ...(4)

Since R.H.S. of Equations (2) and (4) are same.

$\therefore$ L.H.S. also.

$\therefore \sin 3A = 3\sin A - 4\sin^3 A$ ✓ **...Hence verified.**

Ex. 5.1.24 **(W- 06, 2 Marks)**
If $\sin A = 0.4$, find $\cos 2A$ using multiple angle formula.

☑ **Soln. :**

Given $\sin A = 0.4$ ...(1)

Let's calculate, $\cos 2A$ ...(2)

➢ **Use formula for eqn. (2) :** ... [$\cos 2\theta = 1 - 2\sin^2\theta$]

$\therefore \cos 2A = 1 - 2\sin^2 A$

Using value from Equation (1),

$\cos 2A = 1 - 2(0.4)^2$ (Given : $\sin A = 0.4$)

$= 1 - 2(0.16) = 1 - 0.32 = 0.68$

$\cos 2A = 0.68$ ✓ **...Ans.**

Ex. 5.1.25 **(S-11, 4 Marks)**
Prove that : $\dfrac{\sec 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\tan 8\theta}{\tan 2\theta}$

☑ **Soln. :**

L.H.S. $= \dfrac{\sec 8\theta - 1}{\sec 4\theta - 1}$ ...(1)

➢ **Use for eqn. (1) :** ...$\left(\sec\theta = \dfrac{1}{\cos\theta}\right)$

$$= \frac{\dfrac{1}{\cos 8\theta} - 1}{\dfrac{1}{\cos 4\theta} - 1} = \frac{\left(\dfrac{1-\cos 8\theta}{\cos 8\theta}\right)}{\left(\dfrac{1-\cos 4\theta}{\cos 4\theta}\right)} \quad \text{(by simplification)}$$

$$= \frac{1-\cos 8\theta}{\cos 8\theta} \times \frac{\cos 4\theta}{1-\cos 4\theta} = \frac{1-\cos 8\theta}{1-\cos 4\theta} \times \frac{\cos 4\theta}{\cos 8\theta}$$

$$= \frac{1-\cos(2\times 4\theta)}{1-\cos(2\times 2\theta)} \times \frac{\cos 4\theta}{\cos 8\theta} \quad ...(2)$$

➢ **Use formula for eqn. (2) :** ...$\left[\begin{array}{l}\cos 2\theta = 1 - 2\sin^2\theta \\ \Rightarrow 2\sin^2\theta = 1 - \cos 2\theta \\ \Rightarrow 1 - \cos 2\theta = 2\sin^2\theta\end{array}\right]$

$$= \frac{2\sin^2(4\theta)}{2\sin^2(2\theta)} \times \frac{\cos 4\theta}{\cos 8\theta}$$

$$= \frac{2\sin 4\theta \cdot \sin 4\theta \cdot \cos 4\theta}{2\sin 2\theta \cdot \sin 2\theta \cdot \cos 8\theta}$$

$$= \frac{\sin 4\theta \cdot [2\sin 4\theta \cdot \cos 4\theta]}{2\sin 2\theta \cdot \sin 2\theta \cdot \cos 8\theta} \quad \text{(Rearrangement of numerator)}$$

$$= \frac{\sin(2\times 2\theta)\,[2\sin 4\theta \cdot \cos 4\theta]}{2\sin 2\theta \cdot \sin 2\theta \cdot \cos 8\theta} \quad \text{(Note this step)} \quad ...(3)$$

➢ **Use formula for eqn. (3)**

... [$\sin 2\theta = 2\sin\theta \cdot \cos\theta \Rightarrow 2\sin\theta \cdot \cos\theta = \sin 2\theta$]

$$= \frac{\cancel{2\sin(2\theta)} \cdot \cos(2\theta)\,[\sin(2\times 4\theta)]}{\cancel{2\sin 2\theta} \cdot \sin 2\theta \cdot \cos 8\theta}$$

$$= \frac{\cos 2\theta \cdot \sin 8\theta}{\sin 2\theta \cdot \cos 8\theta} \quad ...(4)$$

➔ **Use for eqn. (4) :** ...$\left[\dfrac{\cos\theta}{\sin\theta} = \cot\theta \text{ and } \dfrac{\sin\theta}{\cos\theta} = \tan\theta\right]$

$= \cot 2\theta \cdot \tan 8\theta$ ...(5)

➔ **Use for eqn. (5) :** ...$\left(\cot\theta = \dfrac{1}{\tan\theta}\right)$

L.H.S. $= \dfrac{\tan 8\theta}{\tan 2\theta}$

L.H.S. = R.H.S. ✓ **...Hence proved.**

Ex. 5.1. 26 **(W- 10, 4 Marks)**
Prove that : $\dfrac{1+\sec 2A}{\tan 2A} = \cot A$

☑ **Soln. :**

L.H.S. $= \dfrac{1+\sec 2A}{\tan 2A}$ ...(1)

➢ **Use formula for eqn. (1) :** ...$\left[\sec\theta = \dfrac{1}{\cos\theta}\right]$

$$= \frac{1 + \dfrac{1}{\cos 2A}}{\tan 2A} \quad ...(2)$$

➢ **Use formula for eqn. (2) :** $\ldots\left[\tan\theta = \frac{\sin\theta}{\cos\theta}\right]$

$$= \frac{\frac{\cos 2A + 1}{\cos 2A}}{\frac{\sin 2A}{\cos 2A}}$$

$$= \frac{\cos 2A + 1}{\sin 2A} = \frac{1 + \cos 2A}{\sin 2A} \text{ (Rearrange terms)} \quad \ldots(3)$$

➢ **Use formula for eqn. (3) :** $\ldots [\cos 2\theta = 2\cos^2\theta - 1]$

$$= \frac{1 + (2\cos^2 A - 1)}{\sin 2A}$$

$$= \frac{1 + 2\cos^2 A - 1}{\sin 2A} \quad \ldots(4)$$

➢ **Use formula for eqn. (4) :** $\ldots[\sin 2\theta = 2\sin\theta\cos\theta]$

$$= \frac{2\cos^2 A}{2\sin A \cdot \cos A}$$

$$= \frac{\cos A}{\sin A} = \cot A$$

L.H.S. = R.H.S. ✓ **...Hence proved.**

Ex. 5.1.27 **(W- 2015, 2 Marks)**

Prove that $\cos A = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)$

☑ **Soln. :**

We know, multiple angle 2A formula as,

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$$

$$= 1 - 2\sin^2 A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

Replace A by $\left(\frac{A}{2}\right)$ in above formula,

$$\cos\left(2 \times \frac{A}{2}\right) = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right) = 2\cos^2\left(\frac{A}{2}\right) - 1$$

$$= 1 - 2\sin^2\left(\frac{A}{2}\right) = \frac{1 - \tan^2\left(\frac{A}{2}\right)}{1 + \tan^2\left(\frac{A}{2}\right)}$$

$$\cos A = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right) = 2\cos^2\left(\frac{A}{2}\right) - 1$$

$$\mathbf{\cos A = 1 - 2\sin^2\left(\frac{A}{2}\right) = \frac{1 - \tan^2\left(\frac{A}{2}\right)}{1 + \tan^2\left(\frac{A}{2}\right)}} \checkmark \quad \ldots\text{Ans.}$$

Ex. 5.1.28 : Prove that : $\tan\theta = \frac{2\tan\left(\frac{\theta}{2}\right)}{1 - \tan^2\left(\frac{\theta}{2}\right)}$

☑ **Soln. :** We know, multiple angle 2θ formula as,

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Replace θ by $\left(\frac{\theta}{2}\right)$ in above formula,

$$\tan\left(2 \times \frac{\theta}{2}\right) = \frac{2\tan\left(\frac{\theta}{2}\right)}{1 - \tan^2\left(\frac{\theta}{2}\right)}$$

$$\mathbf{\tan\theta = \frac{2\tan\left(\frac{\theta}{2}\right)}{1 - \tan^2\left(\frac{\theta}{2}\right)}} \checkmark \quad \ldots\text{Ans.}$$

Ex. 5.1.29 **(S- 06, 2 Marks)**

Prove that : $\frac{\sin\theta}{1 + \cos\theta} = \tan\left(\frac{\theta}{2}\right)$

☑ **Soln. :**

$$\text{L.H.S.} = \frac{\sin\theta}{1 + \cos\theta} \quad \ldots(1)$$

➢ **Use formulae for eqn. (1)**

... [For Numerator, $\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$ and
For Denominator,
$\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1 \Rightarrow 1 + \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)$]

$$= \frac{2\sin\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)}$$

$$\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \tan\left(\frac{\theta}{2}\right)$$

L.H.S. = R.H.S.

$$\mathbf{\frac{\sin\theta}{1 + \cos\theta} = \tan\left(\frac{\theta}{2}\right)} \checkmark \quad \ldots\text{Hence proved.}$$

Chapter Ends...

❑❑❑

Chapter 6 : FACTORIZATION AND DE-FACTORIZATION FORMULAE

Exercise 6.1

Ex. 6.1.1 **W-08, 2 Marks.**

Express as product and evaluate sin 99° – sin 81°

Soln. :

Let's calculate, $\sin \underbrace{99°}_{C} - \sin \underbrace{81°}_{D}$ …(1)

➤ **Use factorization formula for eqn. (1)**

$$\ldots\left[\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\cdot \sin\left(\frac{C-D}{2}\right)\right]$$

$$\sin 99° - \sin 81° = 2\cos\left(\frac{99°+81°}{2}\right)\cdot\sin\left(\frac{99°-81°}{2}\right)$$

$$= 2\cos\left(\frac{180°}{2}\right)\cdot\sin\left(\frac{18°}{2}\right)$$

$\therefore \sin 99° - \sin 81° = 2\cos(90°)\cdot\sin(9°)$ …(2)

➤ **Use standard value for eqn. (2) :** …[cos 90° = 0]

$\therefore \sin 99° - \sin 81° = 2(0)\cdot\sin(9°)$

sin 99° – sin 81° = 0 ✓ …**Ans.**

Ex. 6.1.2 **(W-2016, Q. 4(c), W-22, 4 Marks)**

Prove that $\sin 10°\cdot\sin 30°\cdot\sin 50°\cdot\sin 70° = \frac{1}{16}$.

Soln. :

LHS = $\sin 10°\cdot\sin 30°\cdot\sin 50°\cdot\sin 70°$ …(1)

➤ **Use standard value for eqn. (1) :** $\ldots\left[\sin 30° = \frac{1}{2}\right]$

$$= \sin 10°\left(\frac{1}{2}\right)\sin 50°\cdot\sin 70°$$

$$= \frac{1}{2}(\sin 10°\cdot\sin 50°)\cdot\sin 70°$$

$$= \frac{1}{2}\cdot\frac{1}{2}(2\sin\underbrace{50°}_{A}\cdot\sin\underbrace{10°}_{B})\cdot\sin 70°$$

…(Note the adjustment) …(2)

➤ **Use formula for eqn. (2)**

…[2 sin A · sin B = cos (A – B) – cos (A + B)]

$$= \frac{1}{4}[\cos(50°-10°) - \cos(50°+10°)]\cdot\sin 70°$$

$$= \frac{1}{4}[\cos 40° - \cos 60°]\sin 70° \quad \ldots(3)$$

➤ **Use standard value for eqn. (3) :** $\ldots\left[\cos 60° = \frac{1}{2}\right]$

$$= \frac{1}{4}\left[\cos 40° - \frac{1}{2}\right]\sin 70°$$

$$= \frac{1}{4}\sin 70°\left[\cos 40° - \frac{1}{2}\right]$$

$$= \frac{1}{4}\sin 70°\cos 40° - \frac{1}{4}\times\frac{1}{2}\sin 70°$$

$$= \frac{1}{4}\sin 70°\cos 40° - \frac{1}{8}\sin 70°$$

$$= \frac{1}{4}\cdot\frac{1}{2}[2\sin\underbrace{70°}_{A}\cdot\cos\underbrace{40°}_{B}] - \frac{1}{8}\sin 70° \quad \ldots(4)$$

(in 1st term adjustment by 2)

➤ **Use formula for eqn. (4)**

…[2 sin A cos B = sin (A + B) + sin (A – B)]

$$= \frac{1}{8}[\sin(70°+40°) + \sin(70°-40°)] - \frac{1}{8}\sin 70°$$

$$= \frac{1}{8}[\sin(110°) + \underbrace{\sin(30°)}_{\frac{1}{2}}] - \frac{1}{8}\sin 70° \quad \ldots(5)$$

➤ **Use adjustment and standard value for eqn. (5)**

$\ldots\left[110° = 180° - 70° \text{ and } \sin 30° = \frac{1}{2}\right]$

$$= \frac{1}{8}\left[\sin(180°-70°) + \frac{1}{2}\right] - \frac{1}{8}\sin 70°$$

$$= \frac{1}{8}\left[\sin(2\times 90° - 70°) + \frac{1}{2}\right] - \frac{1}{8}\sin 70° \quad \ldots(6)$$

➤ **Use transformation formula for eqn. (6)**

…[sin (2 × 90° – θ) = sin θ]

$$= \frac{1}{8}\left[\sin 70° + \frac{1}{2}\right] - \frac{1}{8}\sin 70°$$

$$= \frac{1}{8}\sin 70° + \frac{1}{16} - \frac{1}{8}\sin 70°$$

L.H.S. $= \frac{1}{16}$ = R.H.S.

$\therefore$ **sin 10° sin 30° sin 50° sin 70° = $\frac{1}{16}$ ✓** …**Hence proved.**

Ex. 6.1.3 S-16, 2 Marks.

Evaluate without using calculator $\frac{\tan 32° + \tan 88°}{1 - \tan 32° \cdot \tan 88°}$

☑ **Soln. :**

Consider, $\frac{\tan 32° + \tan 88°}{1 - \tan 32° \cdot \tan 88°}$ …(1)

➢ **Use formula for eqn. (1)**

$$\ldots\left[\frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} = \tan (A + B)\right]$$

$$\frac{\tan 32° + \tan 88°}{1 - \tan 32° \cdot \tan 88°} = \tan (32° + 88°) = \tan 120°$$

$$= \tan (90° + 30°) \quad \ldots(2)$$

➢ **Use formula for eqn. (2) : …** $\left[\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta\right]$

$= -\cot 30°$ …(3)

➢ **Use standard value for eqn. (3) : …** $[\cot 30° = \sqrt{3}]$

$= -\sqrt{3}$

$$\therefore \frac{\tan 32° + \tan 88°}{1 - \tan 32° \cdot \tan 88°} = -\sqrt{3}$$

Ex. 6.1.4 S-13, S-15, 4 Marks.

Prove that : $\frac{\sin 8x - \sin 5x}{\cos 7x + \cos 6x} = \sin x + \cos x \cdot \tan \frac{x}{2}$

☑ **Soln. :**

$$\text{L.H.S.} = \frac{\sin 8x - \sin 5x}{\cos 7x + \cos 6x} \quad \ldots(1)$$

➢ **Use formulae for eqn. (1)**

… [For Numerator,

$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\cdot\sin\left(\frac{C-D}{2}\right)$ and

For Denominator,

$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)$]

$$\text{L.H.S.} = \frac{2\cos\left(\frac{8x+5x}{2}\right)\cdot\sin\left(\frac{8x-5x}{2}\right)}{2\cos\left(\frac{7x+6x}{2}\right)\cdot\cos\left(\frac{7x-6x}{2}\right)}$$

$$= \frac{\cos\left(\frac{13x}{2}\right)\cdot\sin\left(\frac{3x}{2}\right)}{\cos\left(\frac{13x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)} = \frac{\sin\left(\frac{3x}{2}\right)}{\cos\left(\frac{x}{2}\right)}$$

$$= \frac{\sin\left(x + \frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} \quad \left[\text{Note this } \frac{3x}{2} = x + \frac{x}{2}\right] \ldots(2)$$

➢ **Use formula for eqn. (2)**

…[For Numerator, sin (A + B) = sin A · cos B + cos A · sin B]

$$= \frac{\sin x \cdot \cos\left(\frac{x}{2}\right) + \cos x \cdot \sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}$$

$$\text{L.H.S.} = \frac{\sin x \cdot \cos\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} + \frac{\cos x \cdot \sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}$$

$$= \sin x + \cos x \cdot \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \mathbf{\sin x + \cos x \cdot \tan\left(\frac{x}{2}\right)}$$

L.H.S. = R.H.S. ✓ **Hence proved.**

Ex. 6.1.5 W-12, 4 Marks.

Prove that : $\frac{\sin A - \sin 3A}{\sin^2 A - \cos^2 A} = 2 \sin A$

☑ **Soln. :** L.H.S. $= \frac{\sin A - \sin 3A}{\sin^2 A - \cos^2 A}$

$= \frac{-\sin A + \sin 3A}{-\sin^2 A + \cos^2 A}$ [Multiply numerator and denominator by (– 1)]

$$= \frac{\sin 3A - \sin A}{\cos^2 A - \sin^2 A} \quad \ldots(1)$$

➢ **Use formula for eqn. (1)**

… [For Numerator,

$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\cdot\sin\left(\frac{C-D}{2}\right)$

For Denominator, $\cos^2\theta - \sin^2\theta = \cos 2\theta$]

$$= \frac{2\cos\left(\frac{3A+A}{2}\right)\cdot\sin\left(\frac{3A-A}{2}\right)}{\cos 2A}$$

$$= \frac{2\cos\left(\frac{4A}{2}\right)\cdot\sin\left(\frac{2A}{2}\right)}{\cos 2A} = \frac{2\cos 2A \cdot \sin A}{\cos 2A}$$

$= 2 \sin A = $ R.H.S.

L.H.S. = R.H.S.

i.e. $\mathbf{\frac{\sin A - \sin 3A}{\sin^2 A - \cos^2 A} = 2 \sin A}$ ✓

Hence proved.

Ex. 6.1.6 W-15, 4 Marks.

Prove that sin (A + π/6) – sin (A – π/6) = cos A.

☑ **Soln. :**

$$\text{L.H.S.} = \sin\left(A + \frac{\pi}{6}\right) - \sin\left(A - \frac{\pi}{6}\right) \quad \ldots(1)$$

➢ **Use formula for eqn. (1)**

$$\ldots\left[\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\cdot\sin\left(\frac{C-D}{2}\right)\right]$$

$\sin\left(A+\frac{\pi}{6}\right) - \sin\left(A-\frac{\pi}{6}\right)$

$= 2\cos\left[\frac{\left(A+\frac{\pi}{6}\right)+\left(A-\frac{\pi}{6}\right)}{2}\right]\cdot \sin\left[\frac{\left(A+\frac{\pi}{6}\right)-\left(A-\frac{\pi}{6}\right)}{2}\right]$

$= 2\cos\left(\frac{A+\frac{\pi}{6}+A-\frac{\pi}{6}}{2}\right)\cdot\sin\left(\frac{A+\frac{\pi}{6}-A+\frac{\pi}{6}}{2}\right)$

$= 2\cos\left(\frac{2A}{2}\right)\cdot\sin\left(\frac{2\cdot\frac{\pi}{6}}{2}\right) = 2\cos A\cdot\sin\left(\frac{\pi}{6}\right)$

$= 2\cdot\cos A\cdot\left(\frac{1}{2}\right)$ ...(2)

➢ **Use standard value formula for eqn. (2) :**

$\ldots\left[\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}\right]$

$= \cos A$

$$\sin\left(A+\frac{\pi}{6}\right) - \sin\left(A-\frac{\pi}{6}\right) = \cos\theta \checkmark$$

Hence proved.

Ex. 6.1.7 **W-06, 2 Marks.**

If 2 sin 50° cos 70° = sin A – sin B. Find A and B

☑ **Soln. :** Given, sin A – sin B = 2 sin 50° · cos 70°

sin A – sin B = 2 cos 70° · sin 50° ...(1)

➢ **Use formula for eqn. (1)**

$\ldots\left[\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{A-B}{2}\right)\right]$

$\not{2}\cos\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{A-B}{2}\right) = \not{2}\cos 70°\cdot\sin 50°$

By comparing angles of cos and sine, it gives

$\frac{A+B}{2} = 70°$ and $\frac{A-B}{2} = 50°$

$A+B = 2\times 70°$ $\quad A-B = 2\times 50°$

$A+B = 140°$ $\quad A-B = 100°$ ...(2)

$\therefore$ $A+B = 140°$, $A-B = 100°$ } Adding these

$2A = 240°$

$\therefore A = \frac{240°}{2} = 120°$

$\therefore$ **A = 120°**

From Equation (2), $A+B = 140°$

$B = 140° - A = 140° - 120°$

B = 20°

A = 120° and **B = 20°** ✓

Method II

Given, 2 sin 50° cos 70° = sin A – sin B

$2\cos\underbrace{70°}_{C}\sin\underbrace{50°}_{D} = \sin A - \sin B$ ...(1)

➢ **Use formula for eqn. (1)**

...[2 cos C sin D = sin (C + D) – sin (C – D)]

$\therefore$ 2 cos 70° sin 50° = sin (70° + 50°) – sin (70° – 50°)

2 cos 70° · sin 50° = sin (120°) – sin (20°) ...(2)

From Equation (1) and Equation (2)

Sin A – sin B = sin (120°) – sin (20°)

By equating both sides,

A = 120° and B = 20° ✓ **...Ans.**

Ex. 6.1.8 **S-11, 4 Marks.**

Prove that : $\frac{\sin 8\theta\cdot\cos\theta - \cos 3\theta\cdot\sin 6\theta}{\cos 2\theta\cdot\cos\theta - \sin 3\theta\cdot\sin 4\theta} = \tan 2\theta$

☑ **Soln. :**

$\text{L.H.S.} = \frac{\sin 8\theta\cdot\cos\theta - \cos 3\theta\cdot\sin 6\theta}{\cos 2\theta\cdot\cos\theta - \sin 3\theta\cdot\sin 4\theta}$

$= \frac{2\sin 8\theta\cdot\cos\theta - 2\cos 3\theta\cdot\sin 6\theta}{2\cos 2\theta\cdot\cos\theta - 2\sin 3\theta\cdot\sin 4\theta}$

[Multiply numerator and denominator by 2]

$= \frac{2\sin 8\theta\cdot\cos\theta - 2\sin 6\theta\cdot\cos 3\theta}{2\cos 2\theta\cdot\cos\theta - 2\sin 4\theta\cdot\sin 3\theta}$ (Rearrange) ...(1)

➢ **Use formula for eqn. (1)**

...[For Numerator,
2 sin A · cos B = sin (A + B) + sin (A – B)
For Denominator,
2 cos A · cos B = cos (A + B) + cos (A – B),
2 sin A · sin B = cos (A – B) – cos (A + B)]

$= \frac{[\sin(8\theta+\theta)+\sin(8\theta-\theta)] - [\sin(6\theta+3\theta)+\sin(6\theta-3\theta)]}{[\cos(2\theta+\theta)+\cos(2\theta-\theta)] - [\cos(4\theta-3\theta)-\cos(4\theta+3\theta)]}$

$\text{L.H.S.} = \frac{(\sin 9\theta + \sin 7\theta) - (\sin 9\theta + \sin 3\theta)}{(\cos 3\theta + \cos\theta) - (\cos\theta - \cos 7\theta)}$

$= \frac{\cancel{\sin 9\theta} + \sin 7\theta - \cancel{\sin 9\theta} - \sin 3\theta}{\cos 3\theta + \cancel{\cos\theta} - \cancel{\cos\theta} + \cos 7\theta}$

$= \frac{\sin 7\theta - \sin 3\theta}{\cos 3\theta + \cos 7\theta} = \frac{\sin 7\theta - \sin 3\theta}{\cos 7\theta + \cos 3\theta}$ ...(2)

➢ **Use formula for eqn. (2)**

...[For Numerator,
$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\cdot\sin\left(\frac{C-D}{2}\right)$
For Denominator,
$\cos C + \cos D = 2\cos\left(\frac{C-D}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)$]

$$= \frac{\not{2}\cos\left(\frac{7\theta+3\theta}{2}\right)\cdot\sin\left(\frac{7\theta-3\theta}{2}\right)}{\not{2}\cos\left(\frac{7\theta+3\theta}{2}\right)\cdot\cos\left(\frac{7\theta-3\theta}{2}\right)}$$

$$= \frac{\cos\left(\frac{10\theta}{2}\right)\cdot\sin\left(\frac{4\theta}{2}\right)}{\cos\left(\frac{10\theta}{2}\right)\cdot\cos\left(\frac{4\theta}{2}\right)} = \frac{\cancel{\cos 5\theta}\cdot\sin 2\theta}{\cancel{\cos 5\theta}\cdot\cos 2\theta}$$

$$= \frac{\sin 2\theta}{\cos 2\theta}$$

L.H.S. $= \tan 2\theta$

L.H.S. = R.H.S. ✓ **Hence proved.**

Ex. 6.1.9 **(W-10, 4 Marks)**

Prove that : $\frac{\sin A + 2\sin 2A + \sin 3A}{\cos A + 2\cos 2A + \cos 3A} = \tan 2A$

☑ **Soln. :** L.H.S. $= \frac{\sin A + 2\sin 2A + \sin 3A}{\cos A + 2\cos 2A + \cos 3A}$

$$= \frac{\sin A + \sin 2A + \sin 2A + \sin 3A}{\cos A + \cos 2A + \cos 2A + \cos 3A} \quad \text{[Note this step]}$$

$$= \frac{(\sin 2A + \sin A) + (\sin 3A + \sin 2A)}{(\cos 2A + \cos A) + (\cos 3A + \cos 2A)} \quad \left(\begin{matrix}\text{Rearrange}\\ \text{the terms}\end{matrix}\right) \quad \text{...(1)}$$

➤ **Use formula for eqn. (1)**

… [For Numerator,

$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)$

For Denominator,

$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)$]

$$= \frac{2\sin\left(\frac{2A+A}{2}\right)\cdot\cos\left(\frac{2A-A}{2}\right) + 2\sin\left(\frac{3A+2A}{2}\right)\times\cos\left(\frac{3A-2A}{2}\right)}{2\cos\left(\frac{2A+A}{2}\right)\times\cos\left(\frac{2A-A}{2}\right) + 2\cos\left(\frac{3A+2A}{2}\right)\times\cos\left(\frac{3A-2A}{2}\right)}$$

$$= \frac{2\sin\left(\frac{3A}{2}\right)\cdot\cos\left(\frac{A}{2}\right) + 2\sin\left(\frac{5A}{2}\right)\cdot\cos\left(\frac{A}{2}\right)}{2\cos\left(\frac{3A}{2}\right)\cdot\cos\left(\frac{A}{2}\right) + 2\cos\left(\frac{5A}{2}\right)\cdot\cos\left(\frac{A}{2}\right)}$$

$$= \frac{\cancel{2\cos\left(\frac{A}{2}\right)}\left[\sin\left(\frac{3A}{2}\right) + \sin\left(\frac{5A}{2}\right)\right]}{\cancel{2\cos\left(\frac{A}{2}\right)}\left[\cos\left(\frac{3A}{2}\right) + \cos\left(\frac{5A}{2}\right)\right]}$$

[common out $2\cos\left(\frac{A}{2}\right)$ from numerator, and denominator]

$$= \frac{\sin\left(\frac{3A}{2}\right) + \sin\left(\frac{5A}{2}\right)}{\cos\left(\frac{3A}{2}\right) + \cos\left(\frac{5A}{2}\right)}$$

$$= \frac{\sin\left(\frac{5A}{2}\right) + \sin\left(\frac{3A}{2}\right)}{\cos\left(\frac{5A}{2}\right) + \cos\left(\frac{3A}{2}\right)} \quad \left[\begin{matrix}\text{Rearrange}\\ \text{the terms}\end{matrix}\right] \quad \text{...(2)}$$

➤ **Use formula for eqn. (2)**

… [For Numerator,

$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)$

For Denominator,

$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)$]

$$= \frac{2\sin\left(\frac{\frac{5A}{2}+\frac{3A}{2}}{2}\right)\cdot\cos\left(\frac{\frac{5A}{2}-\frac{3A}{2}}{2}\right)}{2\cos\left(\frac{\frac{5A}{2}+\frac{3A}{2}}{2}\right)\cdot\cos\left(\frac{\frac{5A}{2}-\frac{3A}{2}}{2}\right)}$$

$$= \frac{\not{2}\sin\left[\frac{\left(\frac{5A+3A}{2}\right)}{2}\right]\cdot\cos\left[\frac{\left(\frac{5A-3A}{2}\right)}{2}\right]}{\not{2}\cos\left[\frac{\left(\frac{5A+3A}{2}\right)}{2}\right]\cdot\cos\left[\frac{\left(\frac{5A-3A}{2}\right)}{2}\right]}$$

$$= \frac{\sin\left[\frac{\left(\frac{8A}{2}\right)}{2}\right]\cdot\cos\left[\frac{\left(\frac{2A}{2}\right)}{2}\right]}{\cos\left[\frac{\left(\frac{8A}{2}\right)}{2}\right]\cdot\cos\left[\frac{\left(\frac{2A}{2}\right)}{2}\right]} \quad \text{(by simplification)}$$

$$= \frac{\sin\left(\frac{4A}{2}\right)\cancel{\cdot\cos\left(\frac{A}{2}\right)}}{\cos\left(\frac{4A}{2}\right)\cancel{\cdot\cos\left(\frac{A}{2}\right)}}$$

$$= \frac{\sin(2A)}{\cos(2A)} = \tan 2A$$

∴ L.H.S. = R.H.S. ✓ **Hence proved.**

Chapter Ends...

❑❑❑

Chapter 7 : INVERSE TRIGONOMETRIC FUNCTIONS

Exercise 7.1

Ex. 7.1.1 W-2012, 4 Marks

Prove that : $\sin^{-1} x = \text{cosec}^{-1}\left(\frac{1}{x}\right)$.

Soln. :

Consider, $\sin^{-1} x = \theta$ ...(1)

$\Rightarrow x = \sin\theta$ $[\because \sin^{-1} x = \theta]$ ...(2)

Now, $\text{cosec}^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}\left(\frac{1}{\sin\theta}\right)$ [from Equation (2)]

$\text{cosec}^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}(\text{cosec}\,\theta)$ ...(3)

➤ **Use for eqn. (3) :** ...$[\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta]$

$\therefore \text{cosec}^{-1}\left(\frac{1}{x}\right) = \theta$

$\text{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1} x$ [from Equation (1)]

i.e. $\sin^{-1}x = \text{cosec}^{-1}\left(\frac{1}{x}\right)$ ✓ **...Hence proved.**

Ex. 7.1.2 : Prove that : $\text{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

Soln. :

Consider, $\text{cosec}^{-1} x = \theta$ ...(1)

$x = \text{cosec}\,\theta$ ...(2)

➤ **Use transformation formula for eqn. (2)**

$\ldots\left[\sec\left(\frac{\pi}{2}-\theta\right) = \text{cosec}\,\theta \Rightarrow \text{cosec}\,\theta = \sec\left(\frac{\pi}{2}-\theta\right)\right]$

$\therefore x = \sec\left(\frac{\pi}{2}-\theta\right)$

$\therefore \sec^{-1}(x) = \frac{\pi}{2}-\theta$

$\theta + \sec^{-1} x = \frac{\pi}{2}$

From Equation (1),

$\therefore \text{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$ ✓ **...Hence proved.**

Ex. 7.1.3 S-2011, 2 Marks

Find the value of $\sin\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$

Soln. : Consider $\cos^{-1}\left(-\frac{1}{2}\right) = \theta$ ...(1)

$\therefore -\frac{1}{2} = \cos\theta$, $\frac{1}{2} = -\cos\theta$

i.e. $-\cos\theta = \frac{1}{2}$ ...(2)

➤ **Use transformation formula for eqn. (2)**

...$[\cos(\pi-\theta) = -\cos\theta \Rightarrow -\cos\theta = \cos(\pi-\theta)]$

$\therefore \cos(\pi-\theta) = \frac{1}{2}$ ...(3)

➤ **Use standard value for eqn. (3) :** ...$\left[\cos 60° = \frac{1}{2}\right]$

$\cos(\pi-\theta) = \cos 60°$

By equating,

$\pi - \theta = 60°$

$-\theta = 60° - \pi$

$\theta = \pi - 60°$ ...(4)

$\pi - 60° = \theta$

From Equations (1) and (4)

$\cos^{-1}\left(-\frac{1}{2}\right) = \pi - 60°$

$\therefore \sin\left[\cos^{-1}\left(-\frac{1}{2}\right)\right] = \sin(\pi - 60°)$ ...(5)

➤ **Use for eqn. (5) :** ...$[\sin(\pi-\theta) = \sin\theta]$

$\therefore \sin\left[\cos^{-1}\left(-\frac{1}{2}\right)\right] = \sin 60°$

$\therefore \sin\left[\cos^{-1}\left(-\frac{1}{2}\right)\right] = \frac{\sqrt{3}}{2}$ ✓ (by standard value)

...Ans.

Ex. 7.1.4 W-2006, 2 Marks

Using principal value, find the value of $\cos^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)$

Soln. :

Consider,

$\cos^{-1}\left(\frac{1}{2}\right) = \theta$ ...(1)

$\therefore \frac{1}{2} = \cos\theta$

$\cos\theta = \frac{1}{2}$ ...(3)

$\sin^{-1}\left(\frac{1}{2}\right) = \phi$ ...(2)

$\frac{1}{2} = \sin\phi$

$\sin\phi = \frac{1}{2}$ ...(4)

➤ **Use standard value for eqn. (3) :**

$$\ldots\left[\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}\right]$$

$$\cos\theta = \cos\left(\frac{\pi}{3}\right)$$

By Equating both sides,

$$\theta = \frac{\pi}{3}$$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \quad \text{[From (1)]} \quad \ldots(5)$$

➤ **Use standard value for eqn. (4) :**

$$\ldots\left[\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}\right]$$

$$\sin\phi = \sin\left(\frac{\pi}{6}\right)$$

by Equating both sides

$$\phi = \frac{\pi}{6}$$

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \quad \text{[From (2)]} \quad \ldots(6)$$

From Equations (5) and (6),

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} - \frac{\pi}{6}$$

$$\therefore \mathbf{\cos^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}} \checkmark \quad \textbf{...Ans.}$$

Ex. 7.1.5 **S-2010, 2 Marks.**

Find the principal value of : $\cos^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)$

☑ **Soln. :** Consider,

$$\cos^{-1}\left(\frac{-1}{2}\right) = \theta \quad \ldots(1)$$

$$\frac{-1}{2} = \cos\theta$$

$$\therefore \cos\theta = \frac{-1}{2}$$

$$-\cos\theta = \frac{1}{2} \quad \ldots(3)$$

➤ **Use for eqn. (3)**

$$\ldots[\cos(\pi-\theta) = -\cos\theta]$$

$$\cos(\pi-\theta) = \frac{1}{2} \quad \ldots(5)$$

➤ **Use standard value for eqn. (5) :**

$$\ldots\left[\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}\right]$$

$$\cos(\pi-\theta) = \cos\left(\frac{\pi}{3}\right)$$

By equating both sides,

$$\sin^{-1}\left(\frac{1}{2}\right) = \phi \quad \ldots(2)$$

$$\frac{1}{2} = \sin\phi$$

$$\therefore \sin\phi = \frac{1}{2} \quad \ldots(4)$$

➤ **Use for eqn. (4)**

$$\ldots\left[\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}\right]$$

$$\sin\phi = \sin\left(\frac{\pi}{6}\right)$$

By equating both sides

$$\phi = \frac{\pi}{6}$$

$$\therefore \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\pi - \theta = \frac{\pi}{3} \quad \ldots(6) \text{ [From Equation (2)]}$$

$$-\theta = \frac{\pi}{3} - \pi = \frac{\pi - 3\pi}{3}$$

$$-\theta = \frac{-2\pi}{3}, \quad \therefore \theta = \frac{2\pi}{3}$$

$$\therefore \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3} \quad \text{[From Equation (1)]} \quad \ldots(7)$$

Substituting value from Equations (7) and (6),

$$\text{Now, } \cos^{-1}\left(\frac{-1}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right) = \frac{2\pi}{3} - \frac{\pi}{6}$$

$$= \frac{2\pi(2)-\pi}{6} = \frac{4\pi-\pi}{6} = \frac{3\pi}{6}$$

$$\mathbf{\cos^{-1}\left(\frac{-1}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}} \checkmark \quad \textbf{...Ans.}$$

Ex. 7.1.6 : Prove that : $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$

$$= \tan^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right] = \mathrm{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \cot^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right)$$

☑ **Soln. :** Consider, $\cos\theta = x$ ...(1)

$\Rightarrow \theta = \cos^{-1} x$; i.e. $\cos^{-1} x = \theta$ ...(2)

(I) $\sin^{-1}\left(\sqrt{1-x^2}\right) = \sin^{-1}\left(\sqrt{1-\cos^2\theta}\right)$ [by Equation (1)] ...(3)

➤ **Use for eqn. (3) standard formula**

$$\ldots\left[\begin{array}{l}\sin^2\theta+\cos^2\theta = 1\\ \Rightarrow 1-\cos^2\theta = \sin^2\theta\end{array}\right]$$

$$= \sin^{-1}\left(\sqrt{\sin^2\theta}\right)$$

$$= \sin^{-1}(\sin\theta) \quad \ldots(4)$$

➤ **Use for eqn. (4) :** **...[$\sin^{-1}(\sin\theta) = \theta$]**

$$\sin^{-1}\left(\sqrt{1-x^2}\right) = \theta \quad \ldots(5)$$

(II) $\tan^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right] = \tan^{-1}\left[\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right]$

[by Equation (1)] ...(6)

➤ **Use for eqn. (6)**

$$\ldots\left[\begin{array}{l}\text{Use for Numerator,}\\ \sin^2\theta+\cos^2\theta = 1 \Rightarrow 1-\cos^2\theta = \sin^2\theta\end{array}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{\sin^2\theta}}{\cos\theta}\right] = \tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right]$$

$$= \tan^{-1}(\tan\theta) \quad \ldots(7)$$

➢ **Use for eqn. (7) :** ...[$\tan^{-1}(\tan\theta) = \theta$]

$$\tan^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right] = \theta \quad ...(8)$$

(III) $\text{cosec}^{-1}\left[\frac{1}{\sqrt{1-x^2}}\right] = \text{cosec}^{-1}\left[\frac{1}{\sqrt{1-\cos^2\theta}}\right]$

[by Equation (1)] **...(9)**

➢ **Use for eqn. (9)**

...[For Denominator, $\sin^2\theta + \cos^2\theta = 1 \Rightarrow 1-\cos^2\theta = \sin^2\theta$]

$$= \text{cosec}^{-1}\left[\frac{1}{\sqrt{\sin^2\theta}}\right] = \text{cosec}^{-1}\left[\frac{1}{\sin\theta}\right]$$

$$= \text{cosec}^{-1}(\text{cosec}\,\theta) \quad ...(10)$$

➢ **Use for eqn. (10) :** ...[$\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$]

$$\text{cosec}^{-1}\left[\frac{1}{\sqrt{1-x^2}}\right] = \theta \quad ...(11)$$

(IV) $\cot^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right] = \cot^{-1}\left[\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\right]$

[by Equation (1)] **...(12)**

➢ **Use for eqn. (12)**

...[For Denominator, $\sin^2\theta + \cos^2\theta = 1 \Rightarrow 1-\cos^2\theta = \sin^2\theta$]

$$= \cot^{-1}\left[\frac{\cos\theta}{\sqrt{\sin^2\theta}}\right] = \cot^{-1}\left[\frac{\cos\theta}{\sin\theta}\right]$$

$$= \cot^{-1}(\cot\theta) \quad ...(13)$$

➢ **Use for eqn. (13) :** ...[$\cot^{-1}(\cot\theta) = \theta$]

$$\cot^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right] = \theta \quad ...(14)$$

(V) $\sec^{-1}\left[\frac{1}{x}\right] = \sec^{-1}\left[\frac{1}{\cos\theta}\right]$ [by Equation (1)]

$$= \sec^{-1}(\sec\theta) \quad ...(15)$$

➢ **Use for eqn. (15) :** ...[$\sec^{-1}(\sec\theta) = \theta$]

$$\sec^{-1}\left[\frac{1}{x}\right] = \theta \quad ...(16)$$

R.H.S of Equations (2), (5), (8), (11), (14) and (16) are equal.

∴ L.H.S are also equal, it gives

$$\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right]$$

$$= \text{cosec}^{-1}\left[\frac{1}{\sqrt{1-x^2}}\right] = \cot^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right]$$

$$\therefore \cos^{-1}x = \sec^{-1}\left[\frac{1}{x}\right] \checkmark$$

...Hence proved.

Ex. 7.1.7 W-2015, 4 Marks.

Prove that, $\sin^{-1}x = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$.

☑ **Soln. :** Put, $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$ **...(1)**

$$\sin^{-1}x = \theta \quad ...(2)$$

$$\cot^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right] = \cot^{-1}\left[\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right] \text{(from Equation (1))}$$

$$= \cot^{-1}\left[\frac{\sqrt{\cos^2\theta}}{\sin\theta}\right] \quad (\because \text{ using } \sin^2\theta + \cos^2\theta = 1)$$

$$= \cot^{-1}\left[\frac{\cos\theta}{\sin\theta}\right] = \cot^{-1}(\cot\theta)$$

$$\left(\because \cot\theta = \frac{\cos\theta}{\sin\theta}\right)$$

$$\cot^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right] = \theta \quad \text{(by property)} \quad ...(3)$$

From Equations (2) and (3) [∴R.H.S are equal to L.H.S]

$$\sin^{-1}x = \cot^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right] \checkmark$$

...Hence proved.

Ex. 7.1.8 W-2015, 4 Marks.

Prove that : $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

☑ **Soln. :** Consider, $\tan^{-1}\sqrt{x} = \theta$ **...(1)**

$$\Rightarrow \tan\theta = \sqrt{x} \Rightarrow \tan^2\theta = x \text{ ; i.e. } x = \tan^2\theta \quad ...(2)$$

$$\text{L.H.S} = \tan^{-1}\sqrt{x} = \theta \quad \text{[by Equation (1)]} \quad ...(3)$$

$$\text{R.H.S} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) \quad \text{[by Equation (2)]} \quad ...(4)$$

➢ **Use For eqn. (4) trigonometric formula**

...$\left[\frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta\right]$

$$= \frac{1}{2}\cos^{-1}(\cos 2\theta) \quad ...(5)$$

➢ **Use For eqn. (5) :**

...[$\cos^{-1}(\cos\theta) = \theta$]

$$= \frac{1}{2} \times 2\theta = \theta$$

L.H.S = R.H.S

$$\therefore \tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) \checkmark$$

...Hence proved.

Ex. 7.1.9 : Solve the equation $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$, $x > 0$

☑ **Soln. :**

Consider, $x = \tan\theta$ …(1)

$\Rightarrow \quad \theta = \tan^{-1}x$ …(2)

Given, $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$

$$\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \frac{1}{2}\tan^{-1}(\tan\theta)$$

[From Equation (1)] …(3)

Now consider, $\tan\left(\underbrace{\frac{\pi}{4}}_{A} - \underbrace{\theta}_{B}\right)$ …(4)

➢ **Use trigonometric formula for eqn. (4)**

$$\ldots\left[\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \cdot \tan B}\right]$$

$$\tan\left(\frac{\pi}{4}-\theta\right) = \frac{\tan\frac{\pi}{4} - \tan\theta}{1+\tan\frac{\pi}{4}\cdot\tan\theta} \quad \ldots(5)$$

➢ **Use for eqn. standard value (5) :** $\ldots\left[\tan\left(\frac{\pi}{4}\right) = 1\right]$

$$\tan\left(\frac{\pi}{4}-\theta\right) = \frac{1-\tan\theta}{1+\tan\theta} \quad \ldots(6)$$

Substituting value of Equation (6) in Equation (3),

$$\therefore \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right] = \frac{1}{2}\tan^{-1}(\tan\theta) \quad \ldots(7)$$

θ

➢ **Use for eqn. (7) :** …[$\tan^{-1}(\tan\theta) = \theta$]

$$\frac{\pi}{4} - \theta = \frac{1}{2}\theta$$

$$\frac{\pi}{4} = \frac{1}{2}\theta + \theta$$

$$\frac{1}{2}\theta + \theta = \frac{\pi}{4}$$

$$\frac{3}{2}\theta = \frac{\pi}{4}$$

$$\theta = \frac{\pi}{4}\times\frac{2}{3} = \frac{\pi}{6}, \quad \theta = \frac{\pi}{6}$$

$$\tan^{-1}x = \frac{\pi}{6} \quad \text{[From Equation (2)]}$$

$$x = \tan\left(\frac{\pi}{6}\right) \quad \ldots(8)$$

➢ **Use standard value for eqn. (8) :** $\ldots\left[\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}\right]$

$$\mathbf{x = \frac{1}{\sqrt{3}}} \checkmark$$

…Ans.

Ex. 7.1.10 **S-2013, 4 Marks.**

In x and y are both positive then show that

$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

☑ **Soln. :** Substitute

$x = \tan\theta$ and $y = \tan\phi$ …(1)

$\therefore \theta = \tan^{-1}(x)\ \phi = \tan^{-1}(y)$ …(2)

Now, L.H.S. $= \tan^{-1}x - \tan^{-1}y$

$= \theta - \phi$ [using Equation (2)] …(3)

R.H.S. $= \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

$= \tan^{1}\left(\frac{\tan\theta - \tan\phi}{1+\tan\theta\cdot\tan\phi}\right)$ [using Equation (1)] …(4)

➢ **Use for eqn. (4) :** $\ldots\left[\frac{\tan A - \tan B}{1+\tan A\cdot\tan B} = \tan(A-B)\right]$

$= \tan^{-1}[\tan(\theta-\phi)]$ …(5)

➢ **Use for eqn. (5) :** …[$\tan^{-1}(\tan\theta) = \theta$]

$$\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \theta - \phi \quad \ldots(6)$$

From Equations (3) and (6)

L.H.S. = R.H.S.

$$\therefore \mathbf{\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)} \checkmark$$

…Hence proved.

Ex. 7.1.11 **S-2009, 2 Marks, Q. 3(d), S-18, 4 Marks.**

Prove that : $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) = \cot^{-1}(2)$.

☑ **Soln. :**

Here $x = \frac{1}{4} > 0$ and $y = \frac{2}{9} > 0$

Also, $xy = \frac{1}{4}\cdot\frac{2}{9} = \frac{1}{18} < 1$

L.H.S. $= \tan^{-1}\underbrace{\left(\frac{1}{4}\right)}_{x} + \tan^{-1}\underbrace{\left(\frac{2}{9}\right)}_{y}$ …(1)

➢ **Use standard formula for eqn. (1)**

…[For $x > 0$, $y > 0$ and $xy < 1$, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$]

$\therefore \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) = \tan^{-1}\left[\frac{\frac{1}{4}+\frac{2}{9}}{1-\left(\frac{1}{4}\times\frac{2}{9}\right)}\right]$

$= \tan^{-1}\left[\frac{\frac{9+(2\times 4)}{4\times 9}}{1-\frac{1}{18}}\right]$ (by simplification)

$= \tan^{-1}\left[\frac{\frac{9+8}{36}}{\frac{18-1}{18}}\right] = \tan^{-1}\left[\frac{\left(\frac{17}{36}\right)}{\left(\frac{17}{18}\right)}\right] = \tan^{-1}\left(\frac{17}{36}\times\frac{18}{17}\right)$

$= \tan^{-1}\left(\frac{1}{2}\right)$ …(2)

➢ **Use for eqn. (2) :** …$\left[\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x\right]$

$= \cot^{-1}(2)$

$\therefore$ $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) = \cot^{-1}(2)$ ✓

…Hence proved.

Ex. 7.1.12 S-2010, 4 Marks.

Prove that : $\tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{7}{24}\right) = \cot^{-1}(2)$.

☑ **Soln. :**

Here $x = \frac{2}{11} > 0$ and $y = \frac{7}{24} > 0$

Also, $xy = \frac{2}{11}\cdot\frac{7}{24} = \frac{7}{132} < 1$

L.H.S. $= \tan^{-1}\underbrace{\left(\frac{2}{11}\right)}_{x} + \tan^{-1}\underbrace{\left(\frac{7}{24}\right)}_{y}$ …(1)

➢ **Use for eqn. (1) :**

…$\left[\text{For } x > 0, y > 0 \text{ and } xy < 1,\ \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$\therefore \tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{7}{24}\right) = \tan^{-1}\left[\frac{\frac{2}{11}+\frac{7}{24}}{1-\left(\frac{2}{11}\times\frac{7}{24}\right)}\right]$

$= \tan^{-1}\left[\frac{\frac{48+77}{11\times 24}}{1-\frac{7}{132}}\right] = \tan^{-1}\left[\frac{\left(\frac{125}{264}\right)}{\left(\frac{125}{132}\right)}\right]$

$= \tan^{-1}\left(\frac{125}{264}\times\frac{132}{125}\right)$

$= \tan^{-1}\left(\frac{1}{2}\right)$ …(2)

➢ **Use for eqn. (2) :** …$\left[\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x\right]$

$= \cot^{-1}(2)$

$\therefore \tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{7}{24}\right) = \cot^{-1}(2)$ ✓

…Hence proved.

Ex. 7.1.13 S-2007, 2 Marks.

Prove that : $2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

☑ **Soln. :**

Consider, $x = \frac{1}{3} > 0$

We know, for $x > 0$, $y > 0$ and $xy < 1$

$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left[\frac{x+y}{1-xy}\right]$

$\therefore$ For $x = y$

$\tan^{-1} x + \tan^{-1} x = \tan^{-1}\left[\frac{2x}{1-x^2}\right]$

$2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left[\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right]$ $\left(\because \text{Substitute as } x = \frac{1}{3}\right)$

$2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{\frac{2}{3}}{1-\frac{1}{9}}\right)$ (by simplifications)

$= \tan^{-1}\left(\frac{\frac{2}{3}}{\frac{9-1}{9}}\right) = \tan^{-1}\left[\frac{\left(\frac{2}{3}\right)}{\left(\frac{8}{9}\right)}\right]$

$= \tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right)$

$2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{3}{4}\right)$ ✓ **…Hence Proved**

Ex. 7.1.14 (W-2016, 4 Marks)

Prove that :

$\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

☑ **Soln. :**

L.H.S. $= \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right)$

$= \left[\tan^{-1}\underbrace{\left(\frac{1}{5}\right)}_{x} + \tan^{-1}\underbrace{\left(\frac{1}{7}\right)}_{y}\right] + \left[\tan^{-1}\underbrace{\left(\frac{1}{3}\right)}_{x} + \tan^{-1}\underbrace{\left(\frac{1}{8}\right)}_{y}\right]$ **…(1)**

➢ **Use trigonometric formula for eqn. (1)**

$$\ldots\left[\begin{array}{c}\text{For } x > 0,\ y > 0 \text{ and } xy < 1 \\ \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\end{array}\right]$$

$$= \tan^{-1}\left[\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\cdot\frac{1}{7}}\right] + \tan^{-1}\left[\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\cdot\frac{1}{8}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{7+5}{35}}{1-\frac{1}{35}}\right] + \tan^{-1}\left[\frac{\frac{8+3}{24}}{1-\frac{1}{24}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{12}{35}}{\frac{35-1}{35}}\right] + \tan^{-1}\left[\frac{\frac{11}{24}}{\frac{24-1}{24}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{12}{35}}{\frac{34}{35}}\right] + \tan^{-1}\left[\frac{\frac{11}{24}}{\frac{23}{24}}\right]$$

$$= \tan^{-1}\left(\frac{12}{34}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$

$$= \tan^{-1}\underbrace{\left[\frac{6}{17}\right]}_{x} + \tan^{-1}\underbrace{\left[\frac{11}{23}\right]}_{y} \quad \ldots(2)$$

➢ **Use trigonometric formula for eqn. (2)**

$$\ldots\left[\begin{array}{c}\text{For } x > 0,\ y > 0 \text{ and } xy < 1 \\ \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\end{array}\right]$$

$$= \tan^{-1}\left[\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{16}{17}\cdot\frac{11}{23}}\right] = \tan^{-1}\left[\frac{\frac{138+187}{301}}{1-\frac{66}{391}}\right]$$

$$= \tan^{-1}\left[\frac{325/391}{\left(\frac{391-66}{391}\right)}\right] = \tan^{-1}\left[\frac{\frac{325}{391}}{\frac{325}{391}}\right]$$

$$= \tan^{-1}(1) \quad \ldots(3)$$

➢ **Use for eqn. (3)** $\ldots\left[\tan\frac{\pi}{4} = 1 \Rightarrow \tan-1\,(1) = \frac{\pi}{4}\right]$

$$\text{L.H.S.} = \frac{\pi}{4}$$

$$\therefore \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4} \checkmark$$

...Hence Proved.

Ex. 7.1.15 **S-2013, 4 Marks**

Prove that : $\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \sin^{-1}\left(\frac{84}{85}\right)$

☑ **Soln. :**

Consider,

$$\sin^{-1}\left(\frac{4}{5}\right) = \theta \quad \text{and} \quad \sin^{-1}\left(\frac{8}{17}\right) = \phi \quad \ldots(1)$$

$$\therefore \frac{4}{5} = \sin\theta \qquad \therefore \frac{8}{17} = \sin\phi \quad \ldots(2)$$

i.e. $\sin\theta = \frac{4}{5} = \frac{\text{Opposite side}}{\text{Hypotenuse}}$

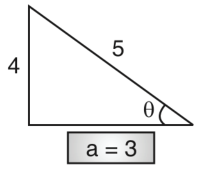

Fig. P. 7.3.48

By Pythagoras triplets

3, 4, 5 **OR**

Suppose third side is a

∴ By Right angle triangle.

$$a^2 + (4)^2 = (5)^2$$
$$a^2 + 16 = 25$$
$$a^2 = 25 - 16 = 9$$
$$\therefore a = \sqrt{9}$$
$$\Rightarrow a = 3$$

$\sin\phi = \frac{8}{17} = \frac{\text{Opposite side}}{\text{Hypotenuse}}$

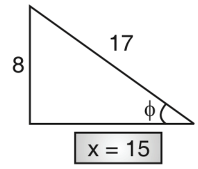

Fig. P. 7.3.48(a)

By Pythagoras triplets

8, 15, 17 **OR**

Suppose third side is x

∴ By Right angle triangle.

$$x^2 + (8)^2 = (17)^2$$
$$x^2 + 64 = 289$$
$$x^2 = 289 - 64$$
$$x^2 = 225$$
$$\Rightarrow x = \sqrt{225} = 15$$

$$\therefore \cos\theta = \frac{3}{5} \quad \ldots(3) \qquad \therefore \cos\phi = \frac{15}{17} \quad \ldots(4)$$

Now,

$$\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \theta + \phi \text{ [from Equation (1)]} \quad \ldots(5)$$

Now consider, $\sin(\underbrace{\theta}_{A} + \underbrace{\phi}_{B})$ …(6)

➢ **For eqn. (6) :**

…[sin (A + B) = sin A · cos B + cos A · sin B]

$$\sin(\theta + \phi) = \sin\theta \cdot \cos\phi + \cos\theta \cdot \sin\phi$$

$$\sin(\theta + \phi) = \left(\frac{4}{5} \times \frac{15}{17}\right) + \left(\frac{3}{5} \times \frac{8}{17}\right)$$

[Values from Equations (2), (4) and (3)]

$$\sin(\theta + \phi) = \frac{12}{17} + \frac{24}{85} = \frac{(12 \times 5) + 24}{85} = \frac{84}{85}$$

$\therefore \; \sin(\theta + \phi) = \dfrac{84}{85}$

$\therefore \; \theta + \phi = \sin^{-1}\left(\dfrac{84}{85}\right)$ ...(7)

Put value of Equation (7) in Equation (5),

$\therefore \; \mathbf{\sin^{-1}\left(\dfrac{4}{5}\right) + \sin^{-1}\left(\dfrac{8}{17}\right) = \sin^{-1}\left(\dfrac{84}{85}\right)}$ ✓

...Hence proved.

Ex. 7.1.16 **W-2012, 4 Marks.**

Show that : $\sin^{-1}\left(\dfrac{3}{5}\right) + \cos^{-1}\left(\dfrac{12}{13}\right) = \cos^{-1}\left(\dfrac{33}{65}\right) = \sin^{-1}\left(\dfrac{56}{65}\right)$

☑ **Soln. :** Consider,

$\sin^{-1}\left(\dfrac{3}{5}\right) = \theta$ and $\cos^{-1}\left(\dfrac{12}{13}\right) = \phi$ ...(1)

$\therefore \; \dfrac{3}{5} = \sin\theta \; ; \; \dfrac{12}{13} = \cos\phi$ ...(2)

$\sin\theta = \dfrac{3}{5} = \dfrac{\text{Opposite side}}{\text{Hypotenuse}}$ $\cos\phi = \dfrac{12}{13} = \dfrac{\text{Adjacent side}}{\text{Hypotenuse}}$

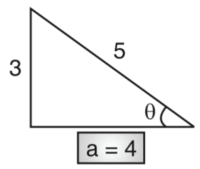

Fig. P. 7.3.49(a)

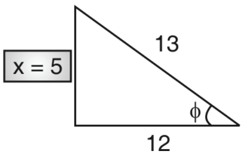

Fig. P. 7.3.49(b)

Using Pythagoras triplets 3, 4, 5 **OR**

Suppose third side is a

$\therefore$ By Right angle triangle,

$a^2 + (3)^2 = (5)^2$

$a^2 + 9 = 25$

$a^2 = 25 - 9$

$\therefore \; a^2 = 16$

$\Rightarrow a = 4$

From Fig. P. 7.3.44(a)

$\therefore \; \cos\theta = \dfrac{4}{5}$ ...(3)

Using Pythagoras triplets 5, 12, 13 **OR**

Suppose third side is x

By Right angle triangle,

$(12)^2 + x^2 = (13)^2$

$144 + x^2 = 169$

$x^2 = 169 - 144$

$x^2 = 25$

$\Rightarrow x = 5$

From Fig. P. 7.3.44(b)

$\therefore \; \sin\phi = \dfrac{5}{13}$ ...(4)

Now, $\sin^{-1}\left(\dfrac{3}{5}\right) + \cos^{-1}\left(\dfrac{12}{15}\right) = \theta + \phi$

[From Equation (1)] ...(5)

(I) Now consider, $\cos(\theta + \phi)$ ...(6)

➢ **Use trigonometric formula for eqn. (6)**

...[cos (A + B) = cos A · cos B – sin A · sin B]

$\cos(\theta + \phi) = \cos\theta \cdot \cos\phi - \sin\theta \cdot \sin\phi$ [By formula]

$\sin(\theta + \phi) = \left(\dfrac{4}{5} \times \dfrac{12}{13}\right) - \left(\dfrac{3}{5} \times \dfrac{5}{13}\right)$

[Values from Equations (3), (2) and (4)]

$= \dfrac{48}{65} - \dfrac{15}{65} = \dfrac{48 - 15}{65} = \dfrac{33}{65}$

$\therefore \; \cos(\theta + \phi) = \dfrac{33}{65}$

$\Rightarrow \; \theta + \phi = \cos^{-1}\left(\dfrac{33}{65}\right)$

Substitute this value in Equation (5)

$\therefore \; \mathbf{\sin^{-1}\left(\dfrac{3}{5}\right) + \cos^{-1}\left(\dfrac{12}{15}\right) = \cos^{-1}\left(\dfrac{33}{65}\right)}$ ✓

...Hence Proved.

(II) Now consider, $\sin(\theta + \phi)$ ...(7)

➢ **Use trigonometric formula for eqn. (8)**

...[sin (A + B) = sin A · cos B + cos A · sin B]

$\therefore \; \sin(\theta + \phi) = \sin\theta \cdot \cos\phi + \cos\theta \cdot \sin\phi$

$= \left(\dfrac{3}{5} \times \dfrac{12}{13}\right) + \left(\dfrac{4}{5} \times \dfrac{5}{13}\right)$

[Values from Equations (2), (3) and (4)]

$= \dfrac{36}{65} + \dfrac{20}{65}$

$\therefore \sin(\theta + \phi) = \dfrac{36 + 20}{65} = \dfrac{56}{65}$

$\therefore \sin(\theta + \phi) = \dfrac{56}{65}$

$\therefore \; \theta + \phi = \sin^{-1}\left(\dfrac{56}{65}\right)$

Substitute this values in Equation (5)

$\therefore \; \mathbf{\sin^{-1}\left(\dfrac{3}{5}\right) + \cos^{-1}\left(\dfrac{12}{15}\right) = \sin^{-1}\left(\dfrac{56}{65}\right)}$ ✓

...Hence Proved.

Ex. 7.1.17 **S-2011, S-2013, 4 Marks.**

Prove that : $\cos^{-1}\left(\dfrac{4}{5}\right) - \sin^{-1}\left(\dfrac{5}{13}\right) = \cos^{-1}\left(\dfrac{63}{65}\right)$

☑ **Soln. :**

Consider,

$\cos^{-1}\left(\dfrac{4}{5}\right) = \theta$ and

$\therefore \; \dfrac{4}{5} = \cos\theta$

$\cos\theta = \dfrac{4}{5}$

$\cos\theta = \dfrac{\text{Adjacent side}}{\text{Hypotenuse}}$

In Right angle triangle

$\sin^{-1}\left(\dfrac{5}{13}\right) = \phi$ ...(1)

$\dfrac{5}{13} = \sin\phi$

$\sin\phi = \dfrac{5}{13}$ ...(2)

$\sin\phi = \dfrac{\text{Opposite side}}{\text{Hypotenuse}}$

In Right angle triangle

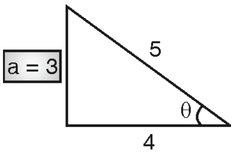

Fig. P. 7.3.51(a)

By Pythagoras triplets

3, 4, 5 **OR**

Suppose third side is a

∴ By Right angle triangle

$\therefore \quad (4)^2 + a^2 = (5)^2$

$16 + a^2 = 25$

$a^2 = 25 - 16$

$a^2 = 9$

$a = \sqrt{9} = 3$

From Fig. P. 7.3.51(a)

$\therefore \quad \sin\theta = \frac{3}{5}$

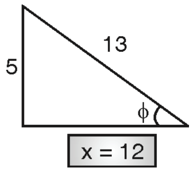

Fig. P. 7.3.51(b)

By Pythagoras triplets

5, 12, 13 **OR**

Suppose third side is x

∴ By Right angle triangle.

$\therefore \quad x^2 + (5)^2 = (13)^2$

$x^2 + 25 = 169$

$x^2 = 169 - 25$

$x^2 = 144$

$x = \sqrt{144} = 12$

From Fig. P. 7.3.51(b)

and $\cos\phi = \frac{12}{13}$ **...(3)**

Now,

$\cos^{-1}\left(\frac{4}{5}\right) - \sin^{-1}\left(\frac{5}{13}\right) = \theta - \phi$ [From Equation (1)] **...(4)**

We know,

$\cos(\theta - \phi) = \cos\theta \cdot \cos\phi + \sin\theta \cdot \sin\phi$ ($\because$ by formula)

$\cos(\theta - \phi) = \left(\frac{4}{5} \times \frac{12}{13}\right) + \left(\frac{3}{5} \times \frac{5}{13}\right) = \frac{48}{65} + \frac{15}{65}$

$= \frac{48 + 15}{65} = \frac{63}{65}$

$\therefore \cos(\theta - \phi) = \frac{63}{65}$

$\therefore \quad \theta - \phi = \cos^{-1}\left(\frac{63}{65}\right)$

Substitute this value in Equation (4), it gives,

$\therefore \boldsymbol{\cos^{-1}\left(\frac{4}{5}\right) - \sin^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{63}{65}\right)}$ ✓

...Hence proved.

Chapter Ends...

❑❑❑

Chapter 8 : STRAIGHT LINE

Exercise 8.1

Ex. 8.1.1 : Find the slope of the line passing through the points (2, 4) and (5, 9)

☑ **Soln. :** We know,

Slope of line passing through the points (x_1, y_1) and (x_2, y_2) is,

$$\mathbf{m = \frac{y_2 - y_1}{x_2 - x_1}} \quad \text{…(1)}$$

Given points, $(x_1, y_1) = (2, 4)$ and $(x_2, y_2) = (5, 9)$

Substitute these values in Equation (1),

$$\therefore \text{Slope of line} = m = \frac{9-4}{5-2} = \frac{5}{3}$$

$$\therefore \quad \textbf{Slope of line} = \frac{5}{3} \checkmark \quad \text{…Ans.}$$

Ex. 8.1.2 : Find the Equation of line with slope $\frac{-3}{2}$ passing through the point (2, – 3)

☑ **Soln. :** We know, **slope point form :**

Equation of line passing through the point (x_1, y_1) with slope m is,

$$\mathbf{y - y_1 = m(x - x_1)} \quad \text{…(1)}$$

Given point $(x_1, y_1) \equiv (2, -3)$

and slope $m = \frac{-3}{2}$

Substitute these values in Equation (1)

$$y - (-3) = \frac{-3}{2}(x - 2)$$

$$y + 3 = \frac{-3}{2}(x - 2)$$

$$2(y + 3) = -3(x - 2)$$

$$2y + 6 = -3x + 6$$

Taking variables on L.H.S. and constants on R.H.S.

$$3x + 2y = 6 - 6$$

$$\mathbf{3x + 2y = 0} \checkmark$$

This is required equation of line.

Ex. 8.1.3 (S-15, 2 Marks)

Find the Equation of line passing through the point (4, – 5) having slope $\frac{-2}{3}$.

☑ **Soln. :** We know, **slope point form :**

Equation of line passing through the point (x_1, y_1) with slope m is,

$$\mathbf{y - y_1 = m(x - x_1)} \quad \text{…(1)}$$

Given point $(x_1, y_1) \equiv (4, -5)$ and slope $m = \frac{-2}{3}$

Substitute these values in Equation (1)

$$y - (-5) = \frac{-2}{3}(x - 4)$$

$$y + 5 = \frac{-2}{3}(x - 4)$$

$$3(y + 5) = -2(x - 4)$$

$$3y + 15 = -2x + 8$$

Taking variables on L.H.S. and constants on R.H.S.

$$2x + 3y = 8 - 15$$

$$\mathbf{2x + 3y = -7} \checkmark$$

This is required equation of line.

Ex. 8.1.4 W-09, 2 Marks

Find the slope and intercepts of 5y = 4 (3 – 2x).

☑ **Soln. :** We know, in a equation of straight line

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{…(1)}$$

Where, X-intercept = a and Y-intercept = b

and $\quad \textbf{Slope of line} = -\dfrac{\textbf{coefficient of x}}{\textbf{coefficient of y}}$

Slope of line : Given, Equation of line is,

$$5y = 4(3 - 2x) = 12 - 8x$$

$$8x + 5y = 12 \quad \text{…(2)}$$

Here coefficient of x = 8 and coefficient of y = 5

$$\therefore \textbf{Slope of line} = -\frac{8}{5}$$

Intercepts

Write Equation (2) in the form of Equation (1),

Divide throughout by 12 to obtain 1 on R.H.S.

$$\frac{8x}{12} + \frac{5y}{12} = \frac{12}{12}$$

$$\frac{2x}{3} + \frac{5y}{12} = 1$$

$$\frac{x}{\left(\frac{3}{2}\right)} + \frac{y}{\left(\frac{12}{5}\right)} = 1 \quad \text{…(3)}$$

Compare with Equation (1)

$$\therefore a = \frac{3}{2}, \quad b = \frac{12}{5}$$

Comparing both

$$\frac{x}{\frac{3}{2}} + \frac{y}{\frac{12}{5}} = 1 \text{ and } \frac{x}{a} + \frac{y}{b}$$

$$\therefore \textbf{X – intercept} = \mathbf{a = \frac{3}{2}}, \quad \textbf{Y – intercept} = \mathbf{b = \frac{12}{5}} \checkmark$$

Ex. 8.1.5 (W-06, S- 10, 2 Marks)

Find the slope and X-intercept of the straight line $\frac{x}{4} - \frac{y}{3} = 2$

☑ **Soln. :** We know, in a equation of straight line

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{…(1)}$$

X-intercept = a ; Y-intercept = b

Given, Equation of line is,

$$\frac{x}{4} - \frac{y}{3} = 2 \quad \text{…(2)}$$

Intercepts

To find intercepts, write Equation (2) in the form of Equation (1),

Divide throughout by 2, to obtain 1 on R.H.S.

$$\frac{x}{4 \times 2} - \frac{y}{3 \times 2} = \frac{2}{2}$$

$$\frac{x}{8} + \frac{y}{-6} = 1$$

Compare with Equation (1)

$\therefore$ a = 8, b = – 6

Comparing equations

$$\frac{x}{8} + \frac{y}{-6} = 1 \text{ and } \frac{x}{a} - \frac{y}{b} = 1$$

$\therefore$ **X - intercept = a = 8 and**

Y - intercept = b = – 6

Slope of line : We know

$$\text{Slope of line} = -\frac{\text{coefficient of x}}{\text{coefficient of y}}$$

Given, Equations is, $\frac{x}{4} - \frac{y}{3} = 2$

$$\left(\frac{1}{4}\right)x + \left(\left(\frac{-1}{3}\right)\right) y = 2$$

↑ coefficient of y ↑ coefficient of y

$$\text{Slope of line} = -\frac{\frac{1}{4}}{\frac{-1}{3}}$$

$$\text{Slope of line} = m = \frac{1}{4} \times \frac{3}{1} = \frac{3}{4}$$

$\therefore$ **Slope of line = m = $\frac{3}{4}$** ✓ **…Ans.**

Ex. 8.1.6 S-07, 4 Marks

The equation of a straight line is 3x – 4y = 12 Find X-intercept and Y-intercept of the line. Also find slope of line.

☑ **Soln. :** We know, in a equation of straight line

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{…(1)}$$

X-intercept = a and Y-intercept = b and

Intercepts : Given, Equation of line is,

$$3x - 4y = 12 \quad \text{…(2)}$$

To find intercepts, write Equation (2) in the form of Equation (1).

Divide throughout by 12, to obtain 1 on R.H.S.

$$\frac{3x}{12} - \frac{4y}{12} = \frac{12}{12}$$

$$\frac{x}{4} - \frac{y}{3} = 1 \quad \text{…(3)}$$

Compare with Equation (1)

$\therefore$ a = 4, b = – 3

$\therefore$ **X-intercept = a = 4 and**

Y-intercept = b = – 3

Slope of line : We know

$$\text{Slope of line} = -\frac{\text{coefficient of x}}{\text{coefficient of y}}$$

Equation is,

$$(3)\, x + (-4)\, y = 12$$

↑ coefficient of x ↑ coefficient of y

$$\text{Slope of line} = -\frac{3}{-4}$$

$\therefore$ **Slope of line = m = $\frac{3}{4}$** ✓

Ex. 8.1.7 (S-11, S-13, 2 Marks)

Show that the lines 2x + 3y – 1 = 0 and 3x – 2y + 6 = 0 are perpendicular to each other.

☑ **Soln. :** We know, two lines having slopes m_1 and m_2 are perpendicular to each other if,

$$m_1 \cdot m_2 = -1.$$

Given Equation of lines are,

▶ **Line I :** 2x + 3y – 1 = 0

$$\therefore \text{Slope of line} = m_1 = -\frac{\text{coefficient of x}}{\text{coefficient of y}}$$

$$(2)\, x + (3)\, y - 1 = 0$$

↑ coefficient of x ↑ coefficient of y

Here coefficient of x = 2 and coefficient of y = 3

$$\text{Slope of line I} = m_1 = \frac{-2}{3} \quad \text{…(1)}$$

Line II :3x – 2y + 6 = 0

$$\therefore \text{Slope of line} = m_2 = -\frac{\text{coefficient of x}}{\text{coefficient of y}}$$

$$(3)\, x - (2)\, y + 6 = 0$$

↑ coefficient of x ↑ coefficient of y

Here coefficient of x = 3 and coefficient of y = – 2

$$\therefore \text{Slope of lien II} = m_2 = -\frac{3}{-2}$$

$$\therefore m_2 = \frac{3}{2} \quad \text{…(2)}$$

Now from Equations (1) and (2)

$$m_1 \times m_2 = \left(\frac{-2}{3}\right) \times \left(\frac{3}{2}\right)$$

$$\mathbf{m_1 \cdot m_2 = -1} \quad \text{[condition of perpendicular lines]}$$

This shows that line, I and line II are perpendicular to each other. ✓

Ex. 8.1.8 **S-15, 2 Marks**

Show that the lines $5x + 6y - 1 = 0$ and $6x - 5y + 3 = 0$ are perpendicular to each other.

☑ **Soln. :** We know, two lines having slopes m_1 and m_2 are perpendicular to each other if

$$m_1 \cdot m_2 = -1.$$

So first find slopes of given lines.

Given Equation of lines are,

- **Line I :** $5x + 6y - 1 = 0$

$\therefore$ Slope of line $= m_1 = -\dfrac{\text{coefficient of x}}{\text{coefficient of y}}$

$$\textcircled{5}\ x + \textcircled{6}\ y - 1 = 0$$

↑ coefficient of x ↑ coefficient of y

Here coefficient of x = 5 and coefficient of y = 6

$$\therefore \text{ Slope of Line I} = m_1 = \frac{-5}{6} \quad \ldots(1)$$

- **Line II :** $6x - 5y + 3 = 0$

$\therefore$ Slope of line $= m_2 = -\dfrac{\text{coefficient of x}}{\text{coefficient of y}}$

Here coefficient of x = 6 and coefficient of y = – 5

$$\therefore \text{ Slope of line II} = m_2 = -\frac{6}{-5}$$

$$\therefore m_2 = \frac{6}{5} \quad \ldots(2)$$

Now from Equations (1) and (2)

$$m_1 \times m_2 = \left(\frac{-5}{6}\right) \times \left(\frac{6}{5}\right)$$

$$\mathbf{m_1 \cdot m_2 = -1} \quad \text{[condition of perpendicular lines]}$$

This shows that line I and line II are perpendicular to each other. ✓

Ex. 8.1.9 **(Q. 5(a)(i), W-17, 6 Marks)**

Find the equation of straight line passes through the points (3, 5) and (4, 6).

☑ **Soln. :** Equation of line is

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

$$\frac{y-5}{5-6} = \frac{x-3}{3-4}$$

$$\frac{y-5}{-1} = \frac{x-3}{-1}$$

$$\mathbf{x - y + 2 = 0} \quad \textbf{…Ans.}$$

Ex. 8.1.10 **W-08, 4 Marks**

Find the equation of line passing through (2, – 3) and parallel to the line $4x - y + 7 = 0$.

☑ **Soln. :** First find slope of given line and use : "Slopes of two parallel lines are same" and slope point from.

Given equation of line is, $4x - y + 7 = 0$ …(1)

$\therefore$ Slope of line $= m = -\dfrac{\text{coefficient of x}}{\text{coefficient of y}}$

$$\textcircled{4}\ x \textcircled{-1}\ y + 6 = 0$$

↑ coefficient of x ↑ coefficient of y

$$\therefore \text{ Slope of line} = m = -\frac{4}{-1} = 4$$

$$m = 4 \quad \ldots(2)$$

We know, slopes of two parallel lines are equal.

Since required line is parallel to the line (1)

$\therefore$ **Slope of required line = m = 4**

[From Equation (2)] …(3)

- **Step II :** Also, we know, Slope-point forms

Equation of line passing through the point (x_1, y_1) and having slope m is,

$$y - y_1 = m(x - x_1) \quad \ldots(4)$$

Required line passing through the point (2, – 3)

Here, $(x_1, y_1) \equiv (2, -3)$

$\therefore x_1 = 2$; $y_1 = -3$

and slope $= m = 4$

Substitute these values in Equation (4),

$$\therefore y - (-3) = 4(x - 2)$$

$$y + 3 = 4(x - 2)$$

$$y + 3 = 4x - 8$$

Taking variables on L.H.S. and constants on R.H.S.

$$-4x + y = -8 - 3$$

$$4x - y = 3 + 8$$

$\mathbf{4x - y = 11}$ ✓ This is required equation of line

Ex. 8.1.11 **S-09, 4 Marks**

Find the equation of the line passing through the intersection of lines $2x - y = 14$ and $2x + y = 10$ and perpendicular to the line $3x - y + 6 = 0$.

☑ **Soln. :**

- **Step I :** First find point of intersection of lines.

Given Equation of lines are,

$$2x - y = 14 \quad \ldots(1)$$

$$2x + y = 10 \quad \ldots(2)$$

Solve the Equations (1) and (2)

$$\left.\begin{aligned} 2x - \cancel{y} &= 14 \\ 2x + \cancel{y} &= 10 \end{aligned}\right\} \text{Adding both equations}$$

$$4x = 24$$

$x = \frac{24}{4}$ $\quad\quad$ $\mathbf{x = 6}$

Put $x = 6$ in Equation (1)

$$2(6) - y = 14$$
$$12 - y = 14$$
$$-y = 14 - 12$$
$$-y = 2 \quad\quad \therefore \mathbf{y = -2}$$
$$\mathbf{x = 6 \quad and \quad y = -2}$$

∴ Point of intersection P is (6, – 2) …(3)

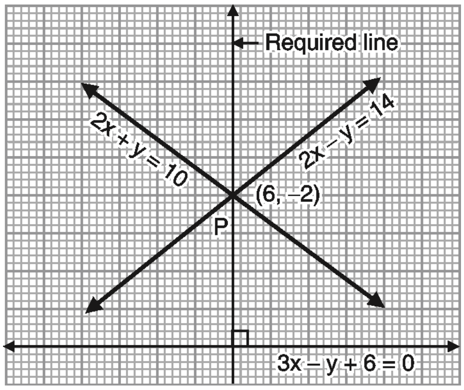

Fig. 8.10.23

- **Step II :** Required line is perpendicular to given line,

Equation of given line is,

$$3x - y + 6 = 0 \quad\quad ...(4)$$

$$\therefore \text{Slope of line} = m_1 = \frac{-\text{(coefficient of x)}}{\text{coefficient of y}}$$

$$3x - 1y + 6 = 0$$

(3) x (–1) y + 6 = 0

↑ coefficient of x ↑ coefficient of y

Here coefficient of x = 3 and coefficient of y = – 1.

$$\therefore \text{Slope of line} = m_1 = \frac{-3}{-1} = 3$$

$$\therefore m_1 = 3 \quad\quad ...(5)$$

We know, if two lines are perpendicular to each other with slopes m_1 and m_2, then,

$$m_1 \cdot m_2 = -1$$

Here required line is perpendicular to line [Equation (4)]

If Slope of required line is m_2, then.

$$m_1 \cdot m_2 = -1 \quad\quad \therefore 3 \times m_2 = -1$$

$$\mathbf{m_2 = \frac{-1}{3}} \quad\quad ...(6)$$

∴ From Equations (5) and (6)

Hence, Required line is passing through the point (6, – 2) and having slope is $\left(\frac{-1}{3}\right)$.

- **Step III :** We know, Slope point form

Equation of line passing through the point (x_1, y_1) having slope m is,

$$y - y_1 = m(x - x_1) \quad\quad ...(7)$$

Here $(x_1, y_1) \equiv (6, -2)$ [Equation (3) point of intersection]

$$\therefore x_1 = 6 \ ; \ y_1 = -2$$

and $m = m_2 = \frac{-1}{3}$ [From Equation (6)]

Substitute these values in Equation (7), it gives,

$$y - (-2) = \left(\frac{-1}{3}\right)(x - 6)$$
$$y + 2 = \frac{-1}{3}(x - 6)$$
$$3(y + 2) = -(x - 6)$$
$$3y + 6 = -x + 6$$

Collecting variables on L.H.S. and constant on R.H.S.

$$x + 3y = 6 - 6$$

$\mathbf{x + 3y = 0}$ ✓ **This is required equation of line**

Ex. 8.1.12 (W-10, 4 Marks)

Find the equation of the line passing through the point of intersection of 2x + y + 6 = 0 and 3x + 5y – 15 = 0 and parallel to the line 5x + 6y + 3 = 0.

☑ **Soln. :**

- **Step I :** First find point of intersection of lines.

Point of intersection of lines,

$$2x + y + 6 = 0 \quad \text{and} \quad 3x + 5y - 15 = 0$$

Rewrite Equation as,

$$2x + y = -6 \quad\quad ...(1)$$
$$3x + 5y = 15 \quad\quad ...(2)$$

To find values of x and y we can use

Method I : Variable cancellation as Ex. 8.10.22. or

Method II : Determinant method

Here we shall use method II

Solve these equations and find values of x and y by determinant method,

$$D = \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} \quad \left[\because \text{Coefficient of x and y of Equations (1)}\right]$$
$$= (2 \times 5) - (1 \times 3) = 10 - 3$$
$$D = 7$$

$$D_x = \begin{vmatrix} -6 & 1 \\ 15 & 5 \end{vmatrix} \quad \left[\text{In D replace coefficient of x by constants of Equations (1)}\right]$$
$$= (-6 \times 5) - (1 \times 15) = -30 - 15$$
$$D_x = -45$$

$$D_y = \begin{vmatrix} 2 & -6 \\ 3 & 15 \end{vmatrix} \quad \left[\text{In D replace coefficient of y by constants of Equations (1)}\right]$$
$$= (2 \times 15) - (-6 \times 3) = 30 + 18$$
$$D_y = 48$$

$\therefore x = \frac{D_x}{D} = \frac{-45}{7}$ and $y = \frac{D_y}{D} = \frac{48}{7}$

$x = \frac{-45}{7}$ and $y = \frac{48}{7}$

$\therefore$ Point of intersection of lines (1) and (2) is $P\left(\frac{-45}{7}, \frac{48}{7}\right)$ …(3)

Step II : Required line is parallel to the line

We know slopes of parallel lines are equal.

$5x + 6y = -3$ …(4)

$\therefore$ slope of given line is,

$$m = -\frac{\text{(coefficient of x)}}{\text{coefficient of y}}$$

$\textcircled{5}\, x + \textcircled{6}\, y + 3 = 0$

coefficient of x, coefficient of y

Here coefficient of x = 5 and coefficient of y = 6

$$m = -\frac{5}{6}$$

Since required line is parallel to line (4)

$\therefore$ Slope of required line $= m = \frac{-5}{6}$ …(5)

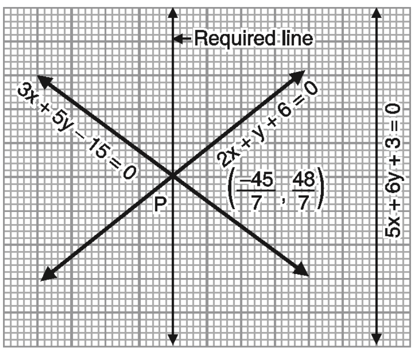

Fig. P. 8.10.26

From equations (3) and (5) required line passing through point $\left(\frac{-45}{7}, \frac{48}{7}\right)$ and having slope is $\left(\frac{-5}{6}\right)$

Step III : We know, slope-point form

Equation of line passing through the point (x_1, y_1) having slope m is,

$$y - y_1 = m(x - x_1) \quad …(6)$$

Here, $P(x_1, y_1) = \left(\frac{-45}{7}, \frac{48}{7}\right)$ and $m = \frac{-5}{6}$

[From Equation (3) and (5)]

Substitute these values in Equation (5)

$$\therefore y - \frac{48}{7} = \left(\frac{-5}{6}\right)\left[x - \left(\frac{-45}{7}\right)\right]$$

$$\frac{7y - 48}{\cancel{7}} = \frac{-5}{6}\,\frac{(7x + 45)}{\cancel{7}}$$

$6(7y - 48) = -5(7x + 45)$ (by cross multiplication)

$42y - 288 = -35x - 225$

Collecting variables on L.H.S. and constants on R.H.S.

$35x + 42y = -225 + 288$

$35x + 42y = 63$

Divide throughout by 7

$\mathbf{5x + 6y = 9}$ ✓

This is required equation of line.

Ex. 8.1.13 (S-16, 4 Marks)

Find the equation of the line passing through the point of intersection of lines x + y = 0 and 2x – y = 9 and point (4, 5).

☑ **Soln. :**

Given : Equation of line are,

$x + y = 0, \quad 2x - y = 9$

$$\left.\begin{aligned} x + \cancel{y} &= 0 \\ 2x - \cancel{y} &= 9 \end{aligned}\right\} \text{ Adding both}$$

$3x = 9$

$\therefore x = 3$

Since $x + y = 0$

$3 + y = 0$

$\therefore y = -3$

$\therefore$ Point of intersection = (3, – 3)

Now we have to find equation of line passing the point of intersection of point (3, – 3) and point (4, 5).

We know, **Two points form :** Equation of line passing through the points, (x_1, y_1) and (x_2, y_2) is,

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Here, $(x_1, y_1) \equiv (3, -3)$ and

$(x_2, y_2) \equiv (4, 5)$

$\therefore \frac{y - (-3)}{5 - (-3)} = \frac{x - 3}{4 - 3}$ …(by cross multiplication)

$\therefore \frac{y + 3}{8} = \frac{x - 3}{1}$

$\therefore y + 3 = 8(x - 3)$

$\therefore y + 3 = 8x - 24$

Collecting variable on L.H.S and constants on R.H.S.

$\therefore -8x + y = -24 - 3$

$-8x + y = -27$

$\therefore \mathbf{8x - y = 27}$ ✓

Which is required equation of line.

Ex. 8.1.14 (W-06, W-11, 4 Marks)

Find the equation of the line passing through (5, 6) and making equal intercepts on the co-ordinate axes.

☑ **Soln. :** We know, **Double intercepts form**

Equation of line having X-intercept a and Y-intercept b is,

$$\frac{x}{a}+\frac{y}{b} = 1 \quad \text{...(1)}$$

Given, Required line making equal intercepts on the co-ordinate axes.

$$\therefore a = b \quad \text{...(2)}$$

Substitute this in Equation (1)

$$\therefore \frac{x}{a}+\frac{y}{a} = 1 \quad \text{...(3)}$$

Since required line is passing through the point (5, 6)

i.e. point $(x_1, y_1) \equiv (5, 6)$ satisfies Equation of line [Equation (3)]

$$\therefore \frac{5}{a}+\frac{6}{a} = 1 \qquad \left[\begin{array}{c}\text{Substitute in Equation (3)}\\ x = 5, y = 6\end{array}\right]$$

$$\frac{5+6}{a} = 1 \qquad \therefore \frac{11}{a} = 1 \Rightarrow \mathbf{a = 11}$$

$$\therefore \mathbf{a = b = 11} \qquad \text{[From Equation (2)]}$$

Substitute these value in Equation (1)

$$\therefore \frac{x}{11}+\frac{y}{11} = 1$$

Multiply throughout by 11,

$\therefore$ **x + y = 11 ✓ Which is equation of required line.**

Ex. 8.1.15 W-07, 4 Marks

Find the equation of line which makes equal and positive intercepts on co-ordinate axes and passing through the point (4, 5).

☑ **Soln. :**

We know, **Double intercepts form**

Equation of line having X-intercept a and Y-intercept b is,

$$\frac{x}{a}+\frac{y}{b} = 1 \quad \text{...(1)}$$

Given, Required line making equal intercepts on the co-ordinate axes.

$$\therefore a = b \quad \text{...(2)}$$

Substitute this in Equation (1)

$$\therefore \frac{x}{a}+\frac{y}{a} = 1 \quad \text{...(3)}$$

Since required line is passing through the point (4, 5)

i.e. point $(x_1, y_1) \equiv (4, 5)$ satisfies Equation of line [Equation (3)]

$$\therefore \frac{4}{a}+\frac{5}{a} = 1 \qquad \left[\begin{array}{c}\text{Substitute in Equation (3)}\\ x = 4, y = 5\end{array}\right]$$

$$\frac{4+5}{a} = 1 \qquad \therefore \frac{9}{a} = 1 \Rightarrow \mathbf{a = 9}$$

$$\therefore \mathbf{a = b = 9} \qquad \text{[From Equation (2)]}$$

Substitute this value in Equation (1)

$$\therefore \frac{x}{9}+\frac{y}{9} = 1$$

Multiply throughout by 9,

$\therefore$ **x + y = 9 ✓ Which is equation of required line.**

Ex. 8.1.16 W-07, W-09, 4 Marks

Find the equation of perpendicular bisector of the line joining the points (4, 8) and (– 2, 6).

☑ **Soln. :**

Perpendicular bisector of line AB means perpendicular to line AB and passing through mid point M.

- **Step I :** Given point, A(4, 8) and B (– 2, 6)

By mid-point formula,

$$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$M = \left(\frac{4-2}{2}, \frac{8+6}{2}\right) \qquad \left[\begin{array}{c}\because x_1 = 4, y_1 = 8 \text{ and}\\ x_2 = -2, y_2 = 6\end{array}\right]$$

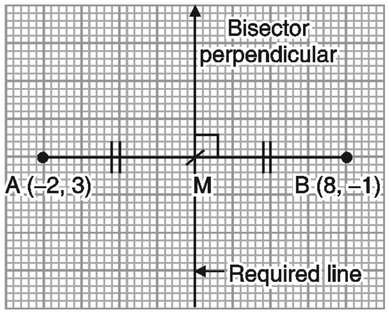

Fig. P. 8.1.16

$$M = \left(\frac{2}{2}, \frac{14}{2}\right) \qquad \therefore M = (1, 7) \quad \text{...(1)}$$

- Step II : Now,

$$\text{Slope of line AB} = \frac{y_2-y_1}{x_2-x_1}$$

Since $(x_1, y_1) \equiv (4, 8)$ and $(x_2, y_2) \equiv (-2, 6)$

$$\text{Slope of line AB} = \frac{6-8}{-2-4}$$

$$\text{Slope of line AB} = \frac{-2}{-6} = \frac{1}{3}$$

$\therefore$ Slope of perpendicular bisector

$$m = \frac{-1}{\text{slope of AB}} \qquad \left[\begin{array}{c}\text{condition of perpendicular}\\ m_1 \cdot m_2 = -1\\ \therefore m_2 = \frac{-1}{m_1}\end{array}\right]$$

$$m = \frac{-1}{\frac{1}{3}} = -3 \quad \text{...(2)}$$

This is slope of required line.

- **Step III :** We know, **Slope point form**

Equation of line passing through the point (x_1, y_1) having slope m is,

$$y - y_1 = m(x - x_1)$$

From equations (1) and (2),

Here, $(x_1, y_1) \equiv (1, 7)$ and slope $= m = -3$

$$y - 7 = -3x + 3 \qquad \therefore 3x + y = 3 + 7$$

$3x + y = 10$ ✓

Which is equation of required line.

Ex. 8.1.17 **S-11, 4 Marks, Q. 5(b)(i), S-18, 3 Marks)**

Find the equation of line passing through the point (3, 4) and perpendicular to the line $2x - 4y + 5 = 0$.

☑ **Soln. :**

▶ **Step I :** Consider slope of required line is m_1.

Given, required line is perpendicular to the line.

$2x - 4y + 5 = 0$ …(1)

Slope of line (1) is,

$$m_2 = \frac{-\text{coefficient of } x}{\text{coefficient of } y}$$

Here coefficient of x = 2

and

coefficient of y = – 4

of given line (1)

$$m_2 = \frac{-2}{-4} = \frac{1}{2}$$

$$\therefore\ m_2 = \frac{1}{2} \quad \ldots(2)$$

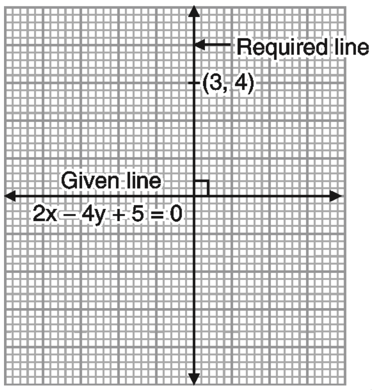

Fig. P. 8.1.17

Since required line and line (1) are perpendicular to each other,

$\therefore\ m_1 \cdot m_2 = -1$ [by condition of perpendicular line]

$m_1 \times \frac{1}{2} = -1$

$m_1 = -1 \times \frac{2}{1}$ $\left[\text{From Equation (2), } m_2 = \frac{1}{2}\right]$

$\mathbf{m_1 = -2}$ …(3)

This is slope of required line

▶ **Step II :** Given line passing through the point (3, 4)

We know, **Slope point form :**

Equation of line passing through point (x_1, y_1) having slope m is

$y - y_1 = m(x - x_1)$ …(4)

The required line passing through the point (3, 4) (given) and having slope $m_1 = -2$ (by Equation 3)

Slope of line $= m_1 = m = -2$ and point $(x_1, y_1) \equiv (3, 4)$

i.e. $x_1 = 3,\ y_1 = 4$

Substitute these value in Equation (4)

$\therefore\ y - 4 = (-2)(x - 3) = -2x + (-2)(-3)$

$y - 4 = -2x + 6$

Collecting variables on L.H.S and contracts R.H.S.

$2x + y = 6 + 4$ [Collect terms of x and y on L.H.S]

$\mathbf{2x + y = 10}$ ✓ **This is required equation of line.**

Ex. 8.1.18 **(Q. 5(b)(i), S-22, 3 Marks)**

Attempt the following : Find equation of line passing through point (2, 0) and perpendicular to $x + y + 3 = 0$

☑ **Soln. :**

▶ **Step I :** Consider slope of required line is m_1.

Given, required line is perpendicular to the line.

$x + y + 3 = 0$ …(1)

Slope of line (1) is,

$$m_2 = \frac{-\text{coefficient of } x}{\text{coefficient of } y}$$

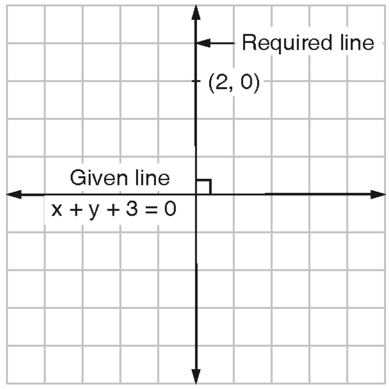

Fig. P. 8.1.18

Here coefficient of x = 1 and coefficient of y = 1 of given line (1)

$m_2 = -\frac{1}{1} = -1 \quad \therefore\ m_1 = 1$ …(2)

Since required line and line (1) are perpendicular to each other,

$\therefore\ m_1 \cdot m_2 = -1$ [by condition of perpendicular line]

$m_1 \times (-1) = -1$

$m_1 = -1 \times (-1)$ $\left[\text{From Equation (2), } m_2 = \frac{1}{2}\right]$

$\mathbf{m_1 = 1}$ …(3)

This is slope of required line

▶ **Step II :** Given line passing through the point (2, 0)

We know, **Slope point form :**

Equation of line passing through point (x_1, y_1) having slope m is

$y - y_1 = m(x - x_1)$ …(4)

The required line passing through the point (2, 0) (given) and having slope $m_1 = 1$ (by Equation 3)

Slope of line $= m_1 = m = 1$ and point $(x_1, y_1) \equiv (2, 0)$

i.e. $x_1 = 2,\ y_1 = 0$

Substitute these value in Equation (4)

$\therefore\ y - 0 = (1)(x - 2) = x - 2$

$y = x - 2$

Collecting variables on L.H.S and contracts R.H.S.

$-x + y = -2$ [Collect terms of x and y on L.H.S]

$\mathbf{x - y = 2}$ ✓ **This is required equation of line.**

Ex. 8.1.19 **S-08, W-10,Q. 5(a)(ii), S-19, 6 Marks)**

Find the equation of a straight line passing through (4, 5) and perpendicular to the line $7x - 5y = 420$.

☑ **Soln. :** To find equation of line, here first find slope of line and use slope point form.

Step I : Consider slope of required line is m_1.

Given, required line is perpendicular to the line.

$$7x - 5y - 420 = 0 \quad \ldots(1)$$

We know, if two lines having slopes m to m_2 are perpendiculars then $m_1 . m_2 = -1$

Slope of line (1) is,

$$m_2 = \frac{-\text{coefficient of } x}{\text{coefficient of } y}$$

Here, coefficient of x = 7 and coefficient of y = – 5.

$$m_2 = \frac{-7}{-5} = \frac{7}{5} \qquad \therefore \ m_2 = \frac{7}{5} \quad \ldots(2)$$

Since required line and line (1) are perpendicular to each other,

$$\therefore \ m_1 \cdot m_2 = -1 \qquad \text{[by condition of, perpendicular line]}$$

$$m_1 \times \frac{7}{5} = -1 \qquad \therefore \ m_1 = -1 \times \frac{5}{7}$$

$$\mathbf{m_1 = \frac{-5}{7}} \quad \ldots(3)$$

This is slope of required line.

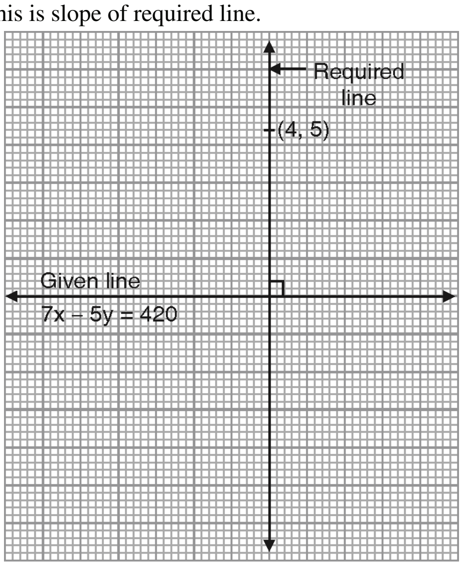

Fig. P. 8.1.19

Step II : Required line passing through the point (4, 5)

We know, **Slope point form**

Equation of line passing through point (x_1, y_1) having slope m is

$$y - y_1 = m (x - x_1) \quad \ldots(4)$$

Here, Slope of required line = $m_1 = m = -\frac{5}{7}$

…(From Equation (3))

and point $(x_1, y_1) \equiv (4, 5)$

i.e. $x_1 = 4, \ y_1 = 5$

Substitute these value in Equation (4)

∴ Equation of Required line is,

$$\therefore \ y - 5 = -\frac{5}{7}(x - 4) \qquad \therefore \ 7(y - 5) = -5(x - 4)$$

$$7y - 35 = -5x + 20$$

Collecting variables on L.H.S. and constants on R.H.S.

$$5x + 7y = 20 + 35$$

$$\mathbf{5x + 7y = 55} \ ✓$$

This is required equation of line.

Ex. 8.1.20 : Triangle ABC has vertices A(4, – 9), B(10, 2) and C(4, – 4). Find the equation of the median from C.

☑ **Soln. :**

A median of a triangle is a line through a vertex and the midpoint of the opposite side. BM is a median of Δ ABC through B.

The vertices of ΔABC are A(4, – 9), B(10, 2) and C(4, – 4)Draw median cm from point C on AB

M is midpoint of AB, By midpoint formula,

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Here

$A = (x_1, y_1) = (4, -9)$ and

$B = (x_2, y_2) = (10, 2)$

$$\therefore M = \left(\frac{4 + 10}{2}, \frac{-9 + 2}{2}\right)$$

$$= \left(\frac{14}{2}, \frac{-7}{2}\right)$$

$$M = \left(7, \frac{-7}{2}\right)$$

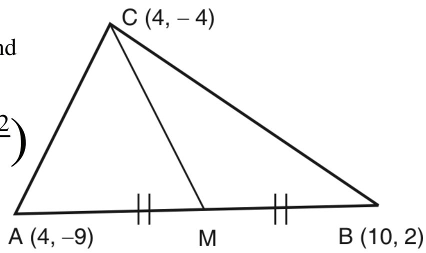

Fig. P. 8.1.20

Now Equation of median CM

We know, two points form :

Equation of line passing through points (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

∴ Equation of median CM

$C = (x_1, y_1) = (4, -4)$

and $M = (x_2, y_2) = \left(7, \frac{-7}{2}\right)$ is

$$\frac{y - (-4)}{\left(\frac{-7}{2}\right) - (-4)} = \frac{x - (4)}{7 - (4)}$$

$$\frac{y + 4}{\frac{-7}{2} + 4} = \frac{x - 4}{3} \qquad \therefore \ \frac{2(y + 4)}{-7 + 8} = \frac{x - 4}{3}$$

$$\frac{2y + 8}{1} = \frac{x - 4}{3}$$

$$3(2y + 8) = x - 4 \qquad \text{(by cross multiplication)}$$

$$6y + 24 = x - 4$$

$$x - 6y = 24 + 4$$

$$\mathbf{x - 6y = 28} \ ✓$$

This is equation of median CM.

Exercise 8.2

Ex. 8.2.1 S-16, 2 Marks

Find the angle between the lines $3x + 2y = 6$ and $2x - 3y = 5$.

Soln. : Let us consider,

$$m_1 = \frac{-3}{2}, \quad m_2 = \frac{2}{3}$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{-3}{2} - \frac{2}{3}}{1 + \frac{-3}{2} \times \frac{2}{3}}\right| = \infty$$

$$\theta = \tan^{-1} \infty = 90° = \frac{\pi}{2} \checkmark$$

Ex. 8.2.2 (W-10, S-11, S-15, 4 Marks)

Find the acute angle between the lines $3x - 2y + 4 = 0$ and $2x - 3y - 7 = 0$.

Soln. :

We know, if θ is the acute angle between two lines with Slopes m_1 and m_2, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \quad \text{…(1)}$$

Given Equation of lines,

- **Line I :** $3x - 2y + 4 = 0$

∴ Slope of line = $m_1 = \frac{-\text{coefficient of } x}{\text{coefficient of } y}$

Here coefficient of x = 3 and coefficient of y = – 2.

$$m_1 = \frac{-3}{-2} \quad \therefore m_1 = \frac{3}{2} \quad \text{…(2)}$$

- **Line II :** $2x - 3y - 7 = 0$

∴ Slope of line = $m_2 = \frac{-\text{coefficient of } x}{\text{coefficient of } y}$

Here coefficient of x = 2 and coefficient of y = – 3.

$$m_2 = \frac{-2}{-3} \quad \therefore m_2 = \frac{2}{3} \quad \text{…(3)}$$

Substitute values of Equations (2) and (3) in Equation (1)

$$\tan\theta = \left|\frac{\frac{3}{2} - \frac{2}{3}}{1 + \left(\frac{3}{2} \times \frac{2}{3}\right)}\right| = \left|\frac{\frac{9-4}{6}}{1+1}\right|$$

$$\tan\theta = \left|\frac{\frac{5}{6}}{2}\right| = \left|\frac{5}{6} \times \frac{1}{2}\right| = \frac{5}{12}$$

$$\theta = \tan^{-1}\left(\frac{5}{12}\right)$$

i.e. $\theta = 22.62°$

$$\theta = 22° 37' \text{ or } \frac{22.62\,\pi^c}{180} \checkmark$$

This is required solution.

Ex. 8.2.3 (Q. 5(b)(i), W-16, 6 Marks)

Find the acute angle between the lines $2x - 3y + 5 = 0$ and $x - 2y - 4 = 0$.

Soln. : We know, if θ is the acute angle between two lines with Slopes m_1 and m_2, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \quad \text{…(1)}$$

Given Equation of lines,

Line I : $2x + 3y + 5 = 0$

∴ Slope of line = $m_1 = \frac{-\text{coefficient of } x}{\text{coefficient of } y}$

Here coefficient of x = 2 and coefficient of y = 3.

$$\text{Slope } m_1 = -\frac{a}{b} = -\frac{2}{3}$$

Line II : $x - 2y - 4 = 0$,

∴ Slope of line = $m_2 = \frac{-\text{coefficient of } x}{\text{coefficient of } y}$

Here coefficient of x = 1

and coefficient of y = – 2

$$\text{Slope } m_2 = -\frac{a}{b} = -\frac{1}{-2} = \frac{1}{2}$$

$$\therefore \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-\frac{2}{3} - \frac{1}{2}}{1 + \left(-\frac{2}{3}\right) \cdot \left(\frac{1}{2}\right)}\right| = \frac{7}{4}$$

$$\therefore \theta = \tan^{-1}\left(\frac{7}{4}\right) \text{ OR } 60.26° \quad \text{…Ans.}$$

Ex. 8.2.4 (W-06, S-10, 4 Marks)

Find the angle between the straight lines $2x + 3y = 13$ and $2x - 5y + 7 = 0$

Soln. : We know,

If θ is the acute angle between two lines with Slopes m_1 and m_2, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \quad \text{…(1)}$$

Given Equation of lines,

- **Line I :** $2x + 3y - 13 = 0$

∴ Slope of line = $m_1 = \frac{-\text{coefficient of } x}{\text{coefficient of } y}$

Here coefficient of x = 2 and coefficient of y = 3.

$$m_1 = \frac{-2}{3} \quad \text{…(2)}$$

- **Line II :** $2x - 5y + 7 = 0$

∴ Slope of line = $m_2 = \frac{-\text{coefficient of } x}{\text{coefficient of } y}$

Here coefficient of x = 2 and coefficient of y = – 5.

$$m_2 = \frac{-2}{-5} \quad \therefore m_2 = \frac{2}{5} \quad \text{…(3)}$$

Substitute values from Equations (2) and (3) in Equation (1).

$$\tan\theta = \left|\frac{\frac{-2}{3}-\frac{2}{5}}{1+\left(\frac{-2}{3}\right)\times\frac{2}{5}}\right| = \left|\frac{\frac{-10-6}{15}}{1-\frac{4}{15}}\right| = \left|\frac{\frac{-16}{\cancel{15}}}{\frac{15-4}{\cancel{15}}}\right| = \left|\frac{-16}{11}\right|$$

$$\tan\theta = \frac{16}{11}$$

$$\theta = \tan^{-1}\left(\frac{16}{11}\right)$$ **This is required solution.**

i.e. $\theta = 55.49°$

$$\theta = 55° \, 29' \text{ or } \frac{55.49\,\pi^c}{180}$$ ✓

Ex. 8.2.5 **W-09, 4 Marks.**

Find the equation of the line passing through $(3, -4)$ and having slope $\frac{3}{2}$.

☑ **Ans. :** We know, slope point form :

Equation of line passing through the point (x_1, y_1) with slope m is,

$$y - y_1 = m(x - x_1) \quad \ldots(1)$$

Given point $(x_1, y_1) \equiv (3, -4)$ and slope $m = \frac{3}{2}$

Substitute these values in Equation (1)

$$y - (-4) = \frac{3}{2}(x-3)$$

$$y + 4 = \frac{3}{2}(x-3)$$

$$2(y+4) = 3(x-3)$$

$$2y + 8 = 3x - 9$$

Taking variables on L.H.S. and constants on R.H.S.

$$-3x + 2y = -9 - 8$$

$$\therefore \; 3x - 2y = -17 = 0 \quad \ldots\textbf{Ans.}$$

This is required equation of line.

Ex. 8.2.6 **S-09, 4 Marks**

Find the perpendicular distance between the point $(3, -2)$ and the line $4x - 6y - 5 = 0$.

☑ **Soln. :**

Given : Point $P(x_1, y_1) = (3, -2)$ …(1)

and line is, $4x - 6y - 5 = 0$

Compare with $ax + by + c = 0$

$a = 4, b = -6, c = -5$ …(2)

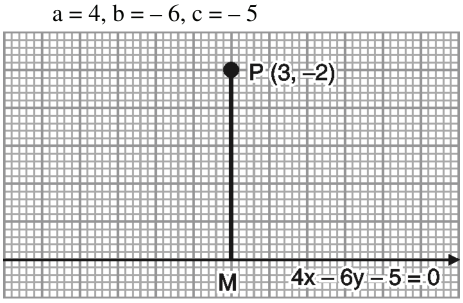

Fig. P. 8.2.6

We know, the length of perpendicular from $P(x_1, y_1)$ on the line $ax + by + c = 0$ is,

$$p = \left|\frac{a x_1 + b y_1 + c}{\sqrt{a^2 + b^2}}\right| = \left|\frac{(4)(3) + (-6)(-2) - 5}{\sqrt{(4)^2 + (-6)^2}}\right|$$

(∵ values from Equations, (1) and (2))

Values from Equations (1) and (2)

$$p = \left|\frac{12 + 12 - 5}{\sqrt{16 + 36}}\right| = \left|\frac{19}{\sqrt{52}}\right| = \frac{19}{\sqrt{52}}$$

$\therefore$ **Perpendicular Distance** $= \frac{19}{\sqrt{52}}$ **Units** ✓

Ex. 8.2.7 **W-08, S-13, 4 Marks.**

Find the length of perpendicular from (3, 4) to the line $3x + 4y - 5 = 0$.

☑ **Soln. :**

Given : Point $P(x_1, y_1) = (3, 4)$ …(1)

and line is, $3x + 4y - 5 = 0$

Compare with $ax + by + c = 0$

$a = 3, b = 4, c = -5$ …(2)

We know, the length of perpendicular from $P(x_1, y_1)$ on the line $ax + by + c = 0$ is,

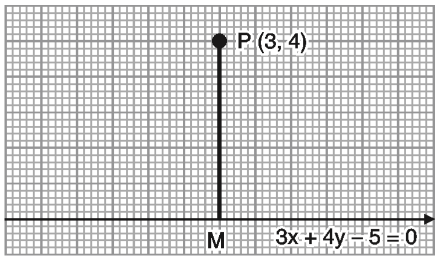

Fig. P. 8.2.7

$$p = \left|\frac{a x_1 + b y_1 + c}{\sqrt{a^2 + b^2}}\right| \quad \left(\because \text{ values from Equations (1) and (2)}\right)$$

$$= \left|\frac{(3)(3) + (4)(4) - 5}{\sqrt{(3)^2 + (4)^2}}\right|$$

Values from Equations (1) and (2)

$$p = \left|\frac{9 + 16 - 5}{\sqrt{9 + 16}}\right| = \left|\frac{20}{\sqrt{25}}\right| = \left|\frac{20}{5}\right| = 4$$

$\therefore$ **Perpendicular Distance = 4 Units** ✓

Ex. 8.2.8 **S-07, 2 Marks**

Find perpendicular distance of a point $\left(0, \frac{5}{4}\right)$ to the line $6x + 8y - 45 = 0$.

☑ **Soln. :**

Given : Point $P(x_1, y_1) = \left(0, \frac{5}{4}\right)$ …(1)

and line is, $6x + 8y - 45 = 0$

Compare with $ax + by + c = 0$

$a = 6, b = 8, c = -45$ …(2)

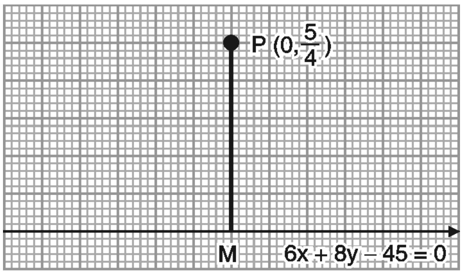

Fig. P. 8.2.8

We know, the length of perpendicular from P (x_1, y_1) on the line $ax + by + c = 0$ is,

$$p = \left|\frac{a x_1 + b y_1 + c}{\sqrt{a^2 + b^2}}\right| = \left|\frac{(6)(0) + (8)\left(\frac{5}{4}\right) - 45}{\sqrt{(6)^2 + (8)^2}}\right|$$

$$p = \left|\frac{0 + 10 - 45}{\sqrt{36 + 64}}\right| = \left|\frac{-35}{\sqrt{100}}\right| = \left|\frac{-35}{10}\right| = \left|\frac{-7}{2}\right|$$

$$p = \frac{7}{2}$$

$\therefore$ **Perpendicular Distance = $\frac{7}{2}$ Units**

Chapter Ends…

❑❑❑

Chapter 9 : FUNCTIONS AND LIMITS

EXERCISE 9.1

Example and Solutions

Ex. 9.1.1 W-2003, S-2014, 4 Marks

If $f(x) = \frac{x-4}{4x-1}$ then show that $f[f(x)] = x$.

Soln. :

Given $f(x) = \frac{x-4}{4x-1}$ …(1)

To prove $f[f(x)] = x$

Replace x by f (x) in Equation (1)

∴ We get,

$$f[f(x)] = \frac{f(x)-4}{4f(x)-1}$$

$$\therefore \quad f[f(x)] = \frac{\left(\frac{x-4}{4x-1}\right)-4}{4\underbrace{\left(\frac{x-4}{4x-1}\right)}-1}$$

$f(x) = \frac{x-4}{4x-1}$ by Equation (1)

$$\therefore \quad f[f(x)] = \frac{\frac{x-4-4(4x-1)}{(4x-1)}}{\frac{4(x-4)-(4x-1)}{(4x-1)}} = \frac{x-4-4(4x-1)}{4(x-4)-(4x-1)}$$

$$= \frac{x-4-16x+4}{4x-16-4x+1} \quad \because -4(4x-1) = -16x+4 \quad \& \quad 4(x-4) = 4x-16$$

Cancelling opposite sign similar terms

$$\Rightarrow \quad f[f(x)] = \frac{x-16x}{-16+1} = \frac{-15x}{-15} = x$$

∴ **$f[f(x)] = x$ ✓ …Hence Proved.**

Ex. 9.1.2 : Test whether the function is even or odd if $f(x) = x^3 + 5\sin x$

Soln. :

Given $f(x) = x^3 + 5\sin x$ …(1)

We know,

If $f(-x) = f(x)$ then f(x) is even function.

and if $f(-x) = -f(x)$ then f(x) is odd function.

Now replace x by (– x) in Equation (1)

We get,

$f(-x) = (-x)^3 + 5\sin(-x)$ …(2)

[∵ $(-x)^3 = -x^3$ as cube of negative term is negative
Also $\sin(-x) = -\sin x$ …by formula of sin angle]

Equation ∴ (2) ⇒ $f(-x) = -x^3 - 5\sin x$

⇒ $f(-x) = -[x^3 + 5\sin x]$ …Taking – ve sign common …(3)

Now, Consider

$-f(x) = -[x^3 + 5\sin x]$ by Equation (1) …(4)

∴ From Equation (3) and (4)

$f(-x) = -f(x)$

∴ **By definition f(x) is odd function. ✓ …Ans.**

Ex. 9.1.3 W- 2011, S-2015, 2 Marks

State whether the function $f(x) = \frac{a^x + a^{-x}}{2}$ is even or odd.

Soln. : Given $f(x) = \frac{a^x + a^{-x}}{2}$ …(1)

We know,

(a) If $f(-x) = f(x)$ then f (x) is even function and

(b) If $f(-x) = -f(x)$ then f (x) is odd function.

Now replace x by – x in Equation (1) we get

$$f(-x) = \frac{a^{-x} + a^{-(-x)}}{2} = \frac{a^{-x} + a^{x}}{2} \quad \{\because (-)(-) = +$$

$$= \frac{a^{-x} + a^{x}}{2} \quad \text{…By commutative law i.e. } a + b = b + a$$

$$f(-x) = \frac{a^{x} + a^{-x}}{2} \quad \text{…(2)}$$

∴ From Equation (1) and (2)

$f(x) = f(-x)$

∴ By definition of even function (i.e. by (a))

Given function f (x) is even function. ✓ …Ans.

EXERCISE 9.2

Ex. 9.2.1 : If $f(x) = \frac{x^3+1}{x^2+1}$, then find f (– 3), f (– 1)

Soln. : Given, $f(x) = \frac{x^3+1}{x^2+1}$ … (1)

Step I : To find f (– 3)

Replace x by – 3 in Equation (1)

We get,

$$f(-3) = \frac{(-3)^3+1}{(-3)^2+1} = \frac{-27+1}{9+1}$$

[∴ on simplification
Numerator & Denominator
∵ $(-3)^3 = -3 \times -3 \times -3$
$9 \times -3 = -27$
and $(-3)^2 = -3 \times -3$
9]

We get,

$$f(-3) = \frac{-26}{10} = \frac{-13}{5}$$

} Divide by 2 on N^r and D^r ∵ both are even number

$$f(-3) = \frac{-13}{5}$$

Step II : To find f (– 1)

Replace x by – 1 in Equation (1)

We get

$$f(-1) = \frac{(-1)^3+1}{(-1)^2+1} = \frac{-1+1}{1+1} = \frac{0}{2}$$

$\left\{ \because (-1)^3 = (-1)(-1)(-1) = -1 \text{ and } (-1)^2 = (-1)(-1) = 1 \right.$

simplifying $\because \frac{0}{a} = 0$ [Here a = 2]

$\therefore$ **f (– 1) = 0** ✓ **…Ans.**

Ex. 9.2.2

S-13, S-16, 4 Marks, W-13, W-15, W-16, 2 Marks

If $f(x) = 16^x + \log_2 x$ then find $f\left(\frac{1}{4}\right)$ and $f\left(\frac{1}{2}\right)$

Also find the value of $f\left(\frac{1}{4}\right)^2$.

☑ **Soln. :**

Given : $f(x) = 16^x + \log_2 x$ …(1)

Step I : To find $f\left(\frac{1}{4}\right)$

Replace x by $\frac{1}{4}$ in Equation (1)

We get,

$$f\left(\frac{1}{4}\right) = \underbrace{(16)^{1/4}}_{[(a)^{1/2}]^{1/2}} + \log_2\left(\frac{1}{4}\right) \quad \left[\because \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} \quad \text{Here } a = 16\right]$$

$$\therefore f\left(\frac{1}{4}\right) = \underbrace{[(16)^{1/2}]^{1/2}}_{\sqrt{16}} + \log_2\left(\frac{1}{4}\right)$$

$\left\{ \because \sqrt{\ } = \frac{1}{2} \quad \therefore (16)^{1/2} = \sqrt{16} \right.$

$$f\left(\frac{1}{4}\right) = \underbrace{[\sqrt{16}\,]^{1/2}}_{4} + \log_2\left(\frac{1}{4}\right)$$

$$f\left(\frac{1}{4}\right) = (4)^{1/2} + \log_2\left(\frac{1}{4}\right)$$

$$= \underbrace{\sqrt{4}}_{2} + \log_2\left(\frac{1}{4}\right)$$

$\left\{ \because (4)^{1/2} = \sqrt{4} \text{ as } \sqrt{\ } = \frac{1}{2} \right.$

$$\therefore \mathbf{f\left(\frac{1}{4}\right) = 2 + \log_2\left(\frac{1}{4}\right)} \quad \text{…(2)}$$

Step II : To find $f\left(\frac{1}{2}\right)$

Replace x by $\frac{1}{2}$ in Equation (1)

We get,

$$f\left(\frac{1}{2}\right) = \underbrace{(16)^{1/2}}_{\sqrt{16}} + \log_2\left(\frac{1}{2}\right)$$

$$f\left(\frac{1}{2}\right) = \underbrace{\sqrt{16}}_{4} + \log_2\left(\frac{1}{2}\right)$$

$$\therefore f\left(\frac{1}{2}\right) = 4 + \log_2\left(\frac{1}{2}\right) \quad [\because \sqrt{16} = 4]$$

$$\therefore f\left(\frac{1}{2}\right) = 4 + \log_2\left(\frac{1}{2}\right) \quad \text{…(3)}$$

Step III : Now, from Equation (2)

Consider $\log_2\left(\frac{1}{4}\right) = y$ …(4)

and from Equation (3)

Consider $\log_2\left(\frac{1}{2}\right) = z$ …(5)

$\therefore$ By Equation (4)

We have $\underbrace{y = \log_2\left(\frac{1}{4}\right)}_{c = \log_a b}$

Taking antilog of both sides,

{ We have , $c = \log_a b \Rightarrow b = a^c$

$$\Rightarrow 2^y = \frac{1}{4}$$

$$\Rightarrow 2^y = \frac{1}{2}\times\frac{1}{2} = \left(\frac{1}{2}\right)^{\mathbf{2}}$$

$$\Rightarrow 2^y = \underbrace{(2^{-1})^2}_{2^{-1\times 2}} \quad \because \frac{1}{a} = a^{-1} \quad \text{Here } a = 2$$

$$\Rightarrow 2^y = 2^{-2}$$

Since base are equal, equating powers from above Equation.

$$y = -2$$

$$\Rightarrow \mathbf{\log_2\left(\frac{1}{4}\right) = -2} \quad \text{…by Equation (4) putting value y} \quad \text{…(6)}$$

Now, by Equation (5)

We have

$$z = \log_2\left(\frac{1}{2}\right)$$

$$\mathbf{c = \log_a (b) \Rightarrow a^c = b}$$

Above Equation becomes

$$2^z = \underbrace{\frac{1}{2}}_{(2)^{-1}} \quad \because \text{Here } a = 2,\ c = z \text{ and } b = \frac{1}{2}$$

as $\frac{1}{a} = a^{-1}$ …by Equation (5)

$$\therefore 2^Z = (2)^{-1}$$

Equating powers we get, (since base are equal)

$$z = -1$$

$$\log_2\left(\frac{1}{2}\right) = -1 \quad \left[\text{by Equation (5) } z = \log_2\left(\frac{1}{2}\right)\right] \quad \ldots(7)$$

Step IV : Now from Equation (2) and (6)

$$f\left(\frac{1}{4}\right) = \underbrace{2 + (-2)}_{2-2=0}$$

$$\therefore \quad f\left(\frac{1}{4}\right) = 0 \checkmark$$

$$\therefore \quad f\left(\frac{1}{4}\right)^2 = \left[f\left(\frac{1}{4}\right)\right]^2 = 0 \checkmark$$

and from Equation (3) and (7)

$$f\left(\frac{1}{2}\right) = \underbrace{4 + (-1)}_{4-1=3}$$

$$\therefore \quad f\left(\frac{1}{2}\right) = 3 \checkmark \quad \ldots\text{Ans.}$$

Ex. 9.2.3 **S-2011, 2 Marks**

If $f(x) = x^3 + x$, find $f(1) + f(2)$

☑ **Soln. :**

Given : $f(x) = x^3 + x$ …(1)

Now, to find f (1)

Replace x by 1 in Equation (1)

$\therefore$ Equation (1) $\Rightarrow$ $f(1) = (1)^3 + 1$

$$(1)^3 \downarrow 1$$

$$= \underbrace{1 + 1}_{2}$$

$$\therefore \quad f(1) = 2 \quad \ldots(2)$$

Now, to find f (2)

Replace x by 2 in Equation (1)

We get,

$\therefore$ Equation (1) $\Rightarrow$ $f(2) = (2)^3 + 2$

$$(2)^3 \downarrow 8 \qquad \{\because\ 2^3 = 2 \times 2 \times 2 = 8$$

$$\therefore \quad f(2) = \underbrace{8 + 2}_{10}$$

$$\therefore \quad f(2) = 10 \quad \ldots(3)$$

$\therefore$ Now to find f (1) + f (2)

Add Equation (2) and (3) both sides

We get,

$$f(1) + f(2) = \underbrace{2 + 10}_{12}$$

$$\therefore \quad \mathbf{f(1) + f(2) = 12} \checkmark \quad \ldots\text{Ans.}$$

Ex. 9.2.4 : If $f(x) = \dfrac{x^2 - 3x + 1}{x - 1}$, find $f(-2) + f\left(\dfrac{1}{3}\right)$

☑ **Soln. :**

Given , $f(x) = \dfrac{x^2 - 3x + 1}{x - 1}$ …(1)

To find $f(-2) + f\left(\dfrac{1}{3}\right)$

We have to find,

$f(-2)$ and $f\left(\dfrac{1}{3}\right)$ separately,

$\therefore$ To find f(–2), replace x by –2 in Equation (1),

We get, $f(-2) = \dfrac{(-2)^2 - 3(-2) + 1}{-2 - 1}$

$$= \frac{4 - (-6) + 1}{-3} \quad \left\{\because\ (-2)^2 = (-2)(-2) = 4 \text{ and } 3(-2) = -6\right\}$$

$$= \frac{4 + 6 + 1}{-3} \quad \{\because\ (-)(-) = +\}$$

$$= \frac{11}{-3}$$

$$\therefore \quad \mathbf{f(-2) = -\frac{11}{3}} \quad \ldots(2)$$

Now, To find $f\left(\dfrac{1}{3}\right)$

Replace x by $\dfrac{1}{3}$ in Equation (1), we get,

We get,

$$\therefore \quad f\left(\frac{1}{3}\right) = \frac{\left(\frac{1}{3}\right)^2 - 3\left(\frac{1}{3}\right) + 1}{\frac{1}{3} - 1}$$

$$= \frac{\frac{1}{9} - \cancel{1} + \cancel{1}}{\frac{1-3}{3}} \quad \left\{\because\ \left(\frac{1}{3}\right)^2 = \frac{1}{9},\ -3\left(\frac{1}{3}\right) = -1,\ \text{Also, } \frac{1}{3} - 1 = \frac{1-3}{3}\right\}$$

$$= \frac{\left(\frac{1}{9}\right)}{\left(\frac{-2}{3}\right)} \quad \left[\therefore \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c}\right]$$

$$= \left(\frac{1}{\cancel{9}_{3}}\right)\left(\frac{\cancel{3}}{-2}\right)$$

$$= \frac{1}{\underbrace{(3)(-2)}_{-6}} = \frac{1}{-6}$$

$$\therefore \quad f\left(\frac{1}{3}\right) = -\frac{1}{6} \quad \ldots(3)$$

$\therefore$ To find $f(-2) + f\left(\dfrac{1}{3}\right)$

Adding Equations (2) and (3),

$\therefore \quad f(-2) + f\left(\frac{1}{3}\right) = \frac{-11}{3} - \frac{1}{6}$

$= \underbrace{\left(\frac{-11}{3} \times \frac{2}{2}\right)}_{} - \frac{1}{6}$

Multiply and Divide by 2 to make denominator 6

$= \frac{(-11)(2)}{6} - \frac{1}{6} = \frac{-22}{6} - \frac{1}{6}$

$= \frac{-22-1}{6} \quad \{\because \; D^r \text{ is same Adding } N^r\}$

$\therefore \quad \mathbf{f(-2) + f\left(\frac{1}{3}\right) = \frac{-23}{6}}$ ✓ **…Ans.**

Ex. 9.2.5 **S-2012, 2 Marks**

If $f(x) = \frac{x^2+9}{\sqrt{x-3}}$, find $f(4) + f(5)$

☑ **Soln. :**

Given, $f(x) = \frac{x^2+9}{\sqrt{x-3}}$ …(1)

To find $f(4) + f(5)$

We have to find f(4) and f(5) separately,

$\therefore$ To find f(4), Replace x by 4 in Equation (1), we get,

$f(4) = \frac{(4)^2+9}{\sqrt{4-3}}$

$= \frac{16+9}{\sqrt{1}} \quad \left\{\because (4)^2 = 4 \times 4 = 16 \text{ and } \sqrt{4-3} = \sqrt{1}\right\}$

$= \frac{25}{1} \quad [\because \sqrt{1} = 1]$

$\therefore \quad \mathbf{f(4) = 25}$ …(2)

Also, To find f(5), Replace x by 5 in Equation (1) we get,

$f(5) = \frac{(5)^2+9}{\sqrt{5-3}}$

$= \frac{25+9}{\sqrt{2}} \quad \left\{\because (5)^2 = 5 \times 5 = 25, \; \sqrt{5-3} = \sqrt{2}\right\}$

$\therefore \quad f(5) = \frac{34}{\sqrt{2}}$

Now, $f(5) = \underbrace{\frac{34}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}}_{}$

Multiplication and Division by $\sqrt{2}$

$\therefore \quad f(5) = \frac{34\sqrt{2}}{\sqrt{2}\sqrt{2}}$

$= \frac{34\sqrt{2}}{2} \quad \left\{\because \sqrt{2}\sqrt{2} = 2 \text{ as } \sqrt{\;\;} = \frac{1}{2}\right\}$

$\therefore \quad f(5) = 17\sqrt{2} \quad \left\{\because \frac{34}{2} = 17\right.$ …(3)

$\therefore$ To find $f(4) + f(5)$

Adding Equation (2) and (3) on both sides we get,

$\mathbf{f(4) + f(5) = 25 + 17\sqrt{2}}$ ✓ **…Ans.**

Ex. 9.2.6 **S-2014, 4 Marks**

If $f(x) = ax^2 + bx + 3$ and $f(1) = 4$, $f(2) = 11$,

Find a and b.

☑ **Soln. :**

Given, $f(x) = ax^2 + bx + 3$ …(1)

$f(1) = 4$ …(2)

and $f(2) = 11$ …(3)

Now replace x by 1 in Equation (1)

We get,

$f(1) = a(1)^2 + b(1) + 3$

$\downarrow$ ***1***

$\Rightarrow \quad 4 = a(1) + b(1) + 3$ …by Equation (2)

$\downarrow$ ***f(1)***

$\Rightarrow \quad 4 = a + b + 3$

$\Rightarrow \quad 4 - 3 = a + b$ …Shifting 3 on left

$\Rightarrow \quad 1 = a + b$

$\therefore \quad a + b = 1 \quad \{\because x = y \Rightarrow y = x\}$ …(4)

Now replace x by 2 in Equation (1)

We get,

$f(2) = a(2)^2 + b(2) + 3$

$\downarrow$ ***4***

$11 = 4a + 2b + 3$ …by Equation (3)

$\downarrow$ ***f(2)***

$\Rightarrow \quad 11 - 3 = 4a + 2b$

$\Rightarrow \quad 8 = 4a + 2b$

$\Rightarrow \quad 4a + 2b = 8$

Divide by 2 to each term we get,

$\Rightarrow \quad \underbrace{\frac{\overset{2}{\cancel{4}}}{\cancel{2}} a}_{2a} + \underbrace{\frac{\cancel{2}}{\cancel{2}} b}_{b} = \frac{\overset{4}{\cancel{8}}}{\cancel{2}}$

$\Rightarrow \quad 2a + b = 4$ …(5)

Now consider Equation (4) and (5)

For solving values of a and b.

$\therefore$ Equation (4) – Equation (5),

We get

$$\begin{array}{rcl} a + \cancel{b} & = & 1 \\ 2a + \cancel{b} & = & 4 \\ - \quad - & & - \\ \hline (a - 2a) & = & (1 - 4) \end{array}$$

{ Substracting Equation (5) from Equation (4) so change sign of each term of Equation (5)

$\Rightarrow \; -a = -3 \qquad \{\because \; 1-4=-3\}$

$\mathbf{a = 3}$

Put this value of a in Equation (4),

We get, $3 + b = 1$ ($\downarrow$ *a*)

$\Rightarrow \; b = 1 - 3$ Shifting 4 to right

$\Rightarrow \; \mathbf{b = -2}$

$\therefore$ **a = 3 ; b = – 2** ✓ **….Ans.**

Ex. 9.2.7 : If $f(x) = x^3 + 1$ and $t = y + 2$ then find f(t).

☑ **Soln. :**

Given, $f(x) = x^3 + 1 \quad \ldots(1)$

and $t = y + 2 \quad \ldots(2)$

$\therefore$ To find f(t), replace x by t in Equation (1), we get,

$\therefore \; f(t) = t^3 + 1$

$\therefore \; f(t) = \underbrace{(y+2)^3}_{(a+b)^3} + 1 \quad \ldots$by (2) $t = y + 2$

$\therefore \; f(t) = \underbrace{[\, y^3 + 3(y^2)(2) + 3(y)(2)^2 + (2)^3\,]}_{a^3 + 3a^2b + 3b^2 + b^3} + 1$

$\{\because \; (a + b^3) = a^3 + 3a^2b + 3ab^2 + b^3\}$

$\therefore \; f(t) = [y^3 + 6y^2 + \underbrace{3(4)}_{12}\, y + 8\,] + 1$

$\{\because \; (3)(2) = 6$ and $(2)^3 = 2 \times 2 \times 2 = 8\}$

$\therefore \; f(t) = y^3 + 6y^2 + 12y + \underbrace{8+1}_{9}$

$\therefore \; \mathbf{f(t) = y^3 + 6y^2 + 12y + 9}$ ✓ **…Ans.**

Ex. 9.2.8 S-2010, 4 Marks

If $y = f(x) = \dfrac{x+1}{x-1}$, $x \neq 1$ then show that, $x = f(y)$.

☑ **Soln. :**

Given, $y = f(x) = \dfrac{x+1}{x-1} \quad \ldots(1)$

$\therefore \; y = \dfrac{x+1}{x-1} \quad \ldots(2)$

and $f(y) = \dfrac{x+1}{x-1} \quad \ldots(3)$

Now show x = f(y)

Put x = y in Equation (3)

We get,

$f(y) = \dfrac{y+1}{y-1}$

But from Equation (2), $y = \dfrac{x+1}{x-1}$

$\therefore \; f(y) = \dfrac{\left(\dfrac{x+1}{x-1}\right) + 1}{\left(\dfrac{x+1}{x-1}\right) - 1}$ Putting value of y.

$\therefore \; f(y) = \dfrac{\dfrac{(x+1)+(x-1)}{(x-1)}}{\dfrac{(x+1)+(x-1)}{(x-1)}}$ Simplifying

$\therefore \; f(y) = \dfrac{(x+1)+(x-1)}{(x-1)-(x-1)} \quad \{\because$ denomination of N^r and D^r is same$\}$

$\therefore \; f(y) = \dfrac{x + 1 + x - 1}{x + 1 - x + 1} \quad \{\because$ On simplication opening brackets and cancelling apposite sign common terms$\}$

$\therefore \; f(y) = \dfrac{x+x}{1+1} = \dfrac{2x}{2} = x$

$\therefore \; \mathbf{f(y) = x}$ ✓**…Hence proved.**

Ex. 9.2.9 W-2007, 4 Marks

If $y = f(x) = \dfrac{x-5}{5x-1}$ then show that $f(y) = x$

☑ **Soln. :**

Given, $y = f(x) = \dfrac{x-5}{5x-1} \quad \ldots(1)$

$y = \dfrac{x-5}{5x-1} \quad \ldots(2)$

and $f(x) = \dfrac{x-5}{5x-1} \quad \ldots(3)$

Now, to prove $f(y) = x$

Replace x by y in Equation (3),

We get, $f(y) = \dfrac{y-5}{5y-1}$

$\therefore \; f(y) = \dfrac{\left(\dfrac{x-5}{5x-1}\right) - 5}{5\left(\dfrac{x-5}{5x-1}\right) - 1} \quad \{\because$ by Equation (2) : $y = \dfrac{x-5}{5x-1}\}$

$\therefore \; f(y) = \dfrac{\dfrac{x-5-5(5x-1)}{5x-1}}{\dfrac{5(x-5)-1(5x-1)}{(5x-1)}}$ Simplifying

$\therefore \; f(y) = \underbrace{\dfrac{x-5-5(5x-1)}{5(x-5)-1(5x-1)}}_{K(a-b)}$

$\therefore \; f(y) = \underbrace{\dfrac{x - 5 - 25x + 5}{5x - 25 - 5x + 1}}_{\text{By scalar multiplication}}$

i.e. $k(a-b) = ka - kb$

$\therefore \; f(y) = \dfrac{x - 25x}{-25 + 1}$

$$= \frac{-24x}{-24} \quad \because x - 25x = -24x$$

$\therefore$ **f(y) = x** ✓ **…Hence Proved.**

Ex. 9.2.10 **W-2008, 2 Marks**

If $f(x) = \frac{x+5}{3x-4}$ and $t = \frac{5+4x}{3x-1}$ show that f(t) = x.

Soln. : Given, $f(x) = \frac{x+5}{3x-4}$ …(1)

and $t = \frac{5+4x}{3x-1}$ …(2)

Now to show f(t) = x

We have to f(t) first

Replace x by t in Equation (1)

We get, $f(t) = \frac{t+5}{3t-4}$

Now, $f(t) = \dfrac{\left(\frac{5+4x}{3x-1}\right)+5}{3\left(\frac{5+4x}{3x-1}\right)-4}$ $\quad \because$ By Equation (2) $t = \left(\frac{5+4x}{3x-1}\right)$

$$= \dfrac{\frac{5+4x+5(3x-1)}{3x-1}}{\frac{3(5+4x)-4(3x-1)}{(3x-1)}} \quad \because \text{ on simplification}$$

$\therefore$ $f(t) = \underbrace{\frac{5+4x+5(3x-1)}{3(5+4x)-4(3x-1)}}_{k(a \pm b)\ form}$

$\therefore$ $f(t) = \underbrace{\frac{5+4x+(15x-5)}{(15+12x)-(12x-4)}}_{k(a\pm b) = ka \pm kb \text{ By scalar multiplication}}$

$\therefore$ $f(t) = \frac{5+4x+15x-5}{15+12x-12x+4}$ {Cancelling Opposite sign common terms}

$\therefore$ $f(t) = \frac{4x+15x}{15+4} = \frac{19x}{19} = x$

$\therefore$ **f(y) = x** ✓ **…Hence proved.**

Ex. 9.2.11 **S-2016, 4 Marks**

If $f(x) = x^2 + 3$ then find the value of x for which f(x) = f(2x+ 1).

Soln. :

Given : $f(x) = x^2 + 3$ …(1)

We have to solve f(x) = f(2x + 1)

Now, $f(2x+1) = \underbrace{(2x+1)^2}_{(a-b)^2} + 3$ …[Replace x by (2x – 1) in Equation (1)]

$= 4x^2 + 4x + 1 + 3$ [(by simplification) $\because (a-b)^2 = a^2 + 2ab + b^2$]

$f(2x+1) = 4x^2 + 4x + 4$ …(2)

Given, f(x) = f(2x + 1)

$x^2 + 3 = 4x^2 + 4x + 4$ …[From Equations (1) and (2)]

$x^2 + 3 - (4x^2 + 4x + 4) = 0$ $\left\{\begin{array}{l}\therefore x = Y \Rightarrow x - y = 0 \\ \text{Here, } X = x^3 + 3 \\ y + 4x^2 - 4x + 4\end{array}\right\}$

$x^2 + 3 - 4x^2 - 4x - 4 = 0$

$(x^2 - 4x^2) - 4x + (3 - 4) = 0$

$-3x^2 - 4x - 1 = 0$

$3x^2 + 4x + 1 = 0$ …[Multiply by (– 1)] …(3)

$3x^2 + 3x + x + 1 = 0$ $\left[\because 3 \times 1 = 3 \text{ and } 3 + 1 = 4\right]$

$(3x^2 + 3x) + (x + 1) = 0$

$3x(x + 1) + (x + 1) = 0$

$(x + 1)(3x + 1) = 0$

$\therefore x + 1 = 0$ OR $3x + 1 = 0$

$x = -1$ $\quad$ $3x = -1$

$x = \frac{-1}{3}$

$$\mathbf{x = -1 \text{ or } x = \frac{-1}{3}} ✓$$ **…Ans.**

Ex. 9.2.12 **S-2008, S-2009, S-2012, 4 Marks**

If f(t) = 50 sin (100 πt + 0.4)

then prove that, $f\left(\frac{1}{50} + t\right) = f(t)$

Soln. : Given, f(t) = 50 sin (100 πt + 0.4) …(1)

Now to prove $f\left(\frac{1}{50} + t\right) = f(t)$

At first we have to find $f\left(\frac{1}{50} + t\right)$

$\therefore$ To find $f\left(\frac{1}{50} + t\right)$

Replace t by $\left(\frac{1}{50} + t\right)$ in Equation (1),

We get,

$f\left(\frac{1}{50} + t\right) = 50 \sin \left[\underbrace{100\pi\left(\frac{1}{50} + t\right)}_{k(a+b)\ form} + 0.4\right]$

$= 50 \sin \left[\underbrace{100\pi\left(\frac{1}{50}\right) + 100\pi t}_{ka + kb \text{ by scalar multiplication}} + 0.4\right]$ $\left\{\begin{array}{l}\because k(a+b) = ka + kb \\ \text{Here } k = 100\pi\end{array}\right\}$

$\therefore$ $f\left(\frac{1}{50} + t\right) = 50 \sin \left[\frac{100\pi}{50} + 100\pi t + 0.14\right]$

$= 50 \sin [\underbrace{2\pi + (100\pi t + 0.4)}_{Sin(2\pi + \theta)}]$

…. Here θ = 100 πt + 0.4

➔ Using trigonometric transformation formula

…[sin (2π + θ) = sin θ]

$\therefore \quad f\left(\frac{1}{50}+t\right) = 50 \sin [100\pi t + 0.4]$

$\therefore \quad f\left(\frac{1}{50}+t\right) = 50 \sin (100\pi t + 0.4)$ …(2)

∴ from Equation (1) and Equation (2)

$f(t) = f\left(\frac{1}{50}+t\right)$

i.e. $f\left(\frac{1}{50}+t\right) = f(t)$ $\{\because x = y \Rightarrow y = x\}$ ✓ **…Hence Proved.**

Ex. 9.2.13 W-2015, 4 Marks

If f(t) = 50 sin (100 πt + 0.04), then show that $f\left(\frac{2}{100}+t\right) = f(t)$

☑ Soln. :

Given : $\frac{2}{100} \Rightarrow \frac{1}{50}$

$f(t) = 50 \sin (100\pi t + 0.04)$ …(1)

We have to prove $f\left(\frac{1}{50}+t\right) = f(t)$

Replace t by $\left(\frac{1}{50}+t\right)$ in Equation (1)

We get,

$\therefore f\left(\frac{1}{50}+t\right) = 50 \sin \left(100\pi\left(\frac{1}{50}+t\right) + 0.04\right)$

…(Simplify bracket)

k (a + b) form

$= 50 \sin \left(100\pi \times \frac{1}{50} + 100\pi \times t + 0.04\right)$

ka + kb by scalar multiplication

$\left\{\because k(a+b) = ka + kb \text{, Here } k = 100\pi\right\}$

$= 50 \sin [2\pi + (100\pi t + 0.04)]$

sin (2π + θ)

Here θ = 100 π + 0.04

➔ Using transformation formula : …[sin (2π + θ) = sin θ]

$\therefore f\left(\frac{1}{50}+t\right) = 50 \sin (100\pi t + 0.04)$

$f\left(\frac{1}{50}+t\right) = f(t)$

$\therefore f\left(\frac{1\times 2}{50\times 2}+t\right) = f(t)$ …[From Equation (1)]

$\therefore f\left(\frac{2}{100}+t\right) = f(t)$ ✓ **…Hence Proved.**

☞ EXERCISE 9.3

Ex. 9.3.1 W-2013, 4 Marks

If $f(x) = \log\left(\frac{x-1}{x+1}\right)$ then prove that $f\left(\frac{x^2+1}{2x}\right) = 2 f(x)$

☑ Soln. :

Given $f(x) = \log\left(\frac{x-1}{x+1}\right)$ …(1)

$\therefore$ To prove, $f\left(\frac{x^2+1}{2x}\right) = 2 f(x)$

We have to find $f\left(\frac{x^2+1}{2x}\right)$ first.

∴ To find $f\left(\frac{x^2+1}{2x}\right)$, replace x by $\left(\frac{x^2+1}{2x}\right)$ in Equation (1) we get,

$$f\left(\frac{x^2+1}{2x}\right) = \log\left[\frac{\left(\frac{x^2+1}{2x}\right)-1}{\left(\frac{x^2+1}{2x}\right)+1}\right]$$

$$= \log\left[\frac{\frac{x^2+1-2x}{2x}}{\frac{x^2+1+2x}{2x}}\right]$$ on simplification

$\left\{\because \frac{a}{b}-1 = \frac{a-b}{b} \text{ and } \frac{a}{b}+1 = \frac{a+b}{b}\right\}$

$\therefore f\left(\frac{x^2+1}{2x}\right) = \log\left(\frac{x^2+1-2x}{x^2+1+2x}\right)$ {… Cancelling common terms in D^r from N^r and D^r}

$= \log\left(\frac{x^2-2x+1}{x^2+2x+1}\right)$ … on Rearranging terms

$\frac{(x-1)^2}{(x+1)^2}$ …on Reduction by factor form

$\therefore f\left(\frac{x^2+1}{2x}\right) = \log\left(\frac{(x-1)^2}{(x+1)^2}\right)$ $\left\{\because (x-1)^2 = x^2-2x+1 \text{ by } (a-b)^2 = a^2-2ab+b^2 \text{ and } (x+1)^2 = x^2+2x+1 \text{ by } (a+b)^2 = a^2+2ab+b^2\right\}$

$\frac{a^m}{b^m}$ *form*

$\therefore f\left(\frac{x^2+1}{2x}\right) = \left(\frac{x-1}{x+1}\right)^2$

$\left(\frac{a}{b}\right)^m$ *form* $\left\{\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m\right\}$

$\therefore f\left(\frac{x^2+1}{2x}\right) = 2 \log\left(\frac{x-1}{x+1}\right)$ …by $\log a^b = b \log a$

f(x) …here $a = \left(\frac{x-1}{x+1}\right)$ and b = 2

$\therefore f\left(\frac{x^2+1}{2x}\right) = 2 f(x)$ ✓ **...Hence Proved.**

$\therefore$ by Equation (1) $f(x) = \log\left(\frac{x-1}{x+1}\right)$

Ex. 9.3.2 **W-2010, W-2012, W-2014, 4 Marks**

If $f(x) = \log\left(\frac{1+x}{1-x}\right)$ then prove that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$

☑ **Soln. :**

Given, $f(x) = \log\left(\frac{1+x}{1-x}\right)$ ...(1)

To prove $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$

We have to find $f\left(\frac{2x}{1+x^2}\right)$ First

$\therefore$ To find $f\left(\frac{2x}{1+x^2}\right)$

Replace x by $\frac{2x}{1+x^2}$ In Equation (1)

We get,

$$f\left(\frac{2x}{1+x^2}\right) = \log\left[\frac{\left(1+\frac{2x}{1+x^2}\right)}{\left(1-\frac{2x}{1+x^2}\right)}\right] \quad \left\} \frac{1+\frac{a}{b}}{1-\frac{a}{b}} \text{ form}\right.$$

$$= \log\left[\frac{\frac{1+x^2+2x}{1+x^2}}{\frac{1+x^2-2x}{1+x^2}}\right] \quad \left\} \because 1+\frac{a}{b} = \frac{b+a}{b} \text{ and } 1-\frac{a}{b} = \frac{b-a}{b}\right.$$

$$\therefore f\left(\frac{2x}{1+x^2}\right) = \log\left[\frac{1+2x+x^2}{1-2x+x^2}\right]$$ ... Cancelling common terms

$$\frac{a^2+2ab+b^2}{a^2-2ab+b^2} = \frac{(a+b)^2}{(a-b)^2} \text{ form}$$

$$= \log\left[\frac{(1+x)^2}{(1-x)^2}\right] \quad \because \text{Here, } a = 1 \text{ and } b = x$$

$$\therefore f\left(\frac{2x}{1+x^2}\right) = \log\left(\frac{1+x}{1-x}\right)^2 \quad \text{...by } \frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$$

➔ **Using Property of Logarithm** ...[$\log a^b = b \log a$]

$$\therefore f\left(\frac{2x}{1+x^2}\right) = \underbrace{2}_{b} \underbrace{\log\left(\frac{1+x}{1-x}\right)}_{a}$$

$$\therefore f\left(\frac{2x}{1+x^2}\right) = 2\underbrace{\log\left(\frac{1+x}{1-x}\right)}_{f(x)}$$

$$\therefore f\left(\frac{2x}{1+x^2}\right) = 2 f(x) \quad \left\{ \text{...by Equation (1) } f(x) = \log\left(\frac{1+x}{1-x}\right) \right\}$$ ✓

...Hence Proved.

Chapter Ends...

❑❑❑

Chapter 10 : DERIVATIVES

Exercise 10.1

Ex. 10.1.1 **S-12, 2 Marks.**

Differentiate w.r.t. x : $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2$

Soln. :

$$y = \underbrace{\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2}_{(a+b)^2\ form} \qquad \dots (1)$$

➔ **Using algebraic formula** …[$(a+b)^2 = a^2+2ab+b^2$]

$$\therefore \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 = (\sqrt{x})^2 + 2(\sqrt{x})\left(\frac{1}{\sqrt{x}}\right)+\left(\frac{1}{\sqrt{x}}\right)^2$$

$$\therefore \text{ Equation (1)} \Rightarrow y = \underbrace{(\sqrt{x})^2}_{x} + 2(\sqrt{x})\left(\frac{1}{\sqrt{x}}\right)+\underbrace{\left(\frac{1}{\sqrt{x}}\right)^2}_{\frac{1}{x}}$$

$$\therefore y = x + \underbrace{2(1)}_{2} + \frac{1}{x} \qquad \dots \text{Simplification}$$

$$\therefore y = x+2+\frac{1}{x}$$

Now, Differentiate both sides w.r.to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x+2+\frac{1}{x}\right)$$

➔ **Using Linearity property of derivative**

$$\dots\left[\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)\right]$$

$$= \underbrace{\frac{d}{dx}(x)}_{1} + \underbrace{\frac{d}{dx}(2)}_{0} + \underbrace{\frac{d}{dx}\left(\frac{1}{x}\right)}_{-1/x^2}$$

$$\therefore \frac{dy}{dx} = 1+0+\left(-\frac{1}{x^2}\right) = 1-\frac{1}{x^2}$$

$$\therefore \frac{dy}{dx} = 1-\frac{1}{x^2} \checkmark \qquad \dots\text{Ans.}$$

Ex. 10.1.2 : Differentiate w.r. to x, $3^x + x^3 + 3^3$

Soln. : Differentiate both sides w.r. to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(3^x+x^3+3^3)$$

➔ **Using derivative of standard function :**

$$\dots\left[\frac{d}{dx}(a^x) = a^x\log a\ ;\ \text{Here } a = 3 \text{ and } \frac{d}{dx}x^n = n\,x^{n-1}\right]$$

$$= \underbrace{\frac{d}{dx}(3^x)}_{3^x\log 3} + \underbrace{\frac{d}{dx}(x^3)}_{3x^{3-1}} \ \underbrace{\frac{d}{dx}(3^3)}_{0}$$

$$\therefore \frac{dy}{dx} = 3^x\log 3 + 3\,x^{3-1}+0$$

$$\therefore \frac{dy}{dx} = 3^x\log 3 + 3x^2 \checkmark \qquad \dots\text{Ans.}$$

Ex. 10.1.3 : If y = sin x + tan x, find $\frac{dy}{dx}$ at $x = \frac{\pi}{3}$

Soln. : Given, y = sin x + tan x

Differentiate this w.r. to x on both sides we get

$$\frac{dy}{dx} = \frac{d}{dx}[\sin x+\tan x]$$

➔ **Using Linearity property of derivative :**

$$\dots\left[\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)\right]$$

$$= \frac{d}{dx}(\sin x)+\frac{d}{dx}(\tan x)$$

➔ **Using derivative of standard function :**

$$\dots\left[\frac{d}{dx}\sin x = \cos x \text{ and } \frac{d}{dx}\tan x = \sec^2 x\right]$$

$$\therefore \frac{dy}{dx} = \cos x + \sec^2 x$$

Now, at $x = \frac{\pi}{3}$ means put $x = \frac{\pi}{3}$.

We get, $$\left(\frac{dy}{dx}\right)_{x=\pi/3} = \underbrace{\cos\left(\frac{\pi}{3}\right)}_{\frac{1}{2}} + \underbrace{\sec^2\left(\frac{\pi}{3}\right)}_{(2)^2}$$

➔ **Using standard values of trigonometric function**

$$\dots\left[\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \text{ and } \sec\left(\frac{\pi}{3}\right) = 2\right]$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=\pi/2} = \frac{1}{2}+\underbrace{(2)^2}_{4}$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=\pi/2} = \frac{1}{2}+4 = \frac{1+(4\times 2)}{2} = \frac{1+8}{2} = \frac{9}{2}$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=\pi/2} = \frac{9}{2} \checkmark \qquad \dots\text{Ans.}$$

Ex. 10.1.4 **W-07, 2 Marks.**

Find $\frac{dy}{dx}$, if $y = a^x x^a$

Soln. :

Given $y = a^x x^a$

Differentiate both sides w.r. to x, we get

$$\frac{dy}{dx} = \underbrace{\frac{d}{dx}(a^x\cdot x^a)}_{\frac{d}{dx}(u\cdot v)\ form} \qquad \dots (1)$$

➔ Using Product rule of derivative :

$$\ldots \left[\frac{d}{dx}(u \cdot v) = u\frac{d}{dx} v + v\frac{d}{dx} u \right]$$

Here $u = a^x$ and $v = x^a$

$$\therefore \text{Equation (1)} \Rightarrow \frac{dy}{dx} = a^x \underbrace{\frac{d}{dx} x^a}_{ax^{a-1}} + x^a \underbrace{\frac{d}{dx} a^x}_{a^x \log a}$$

… By derivative of standard function

$$\therefore \frac{dy}{dx} = a^x \cdot a \underbrace{x^{a-1}}_{x^{a-1} \cdot x} + x^a a^x \log a$$

$$\therefore \frac{dy}{dx} = \underbrace{a^x \cdot a x^{a-1} + x^{a-1} \cdot x\, a^x \log a}_{a^x x^{a-1} [a + x \log a]}$$

… Taking $a^x \cdot x^{a-1}$ common

$$\therefore \frac{dy}{dx} = a^x x^{a-1} [a + x \log a] \checkmark$$ **…Ans.**

Ex. 10.1.5 S-07, 2 Marks.

If $y = (1 + x^2) \tan^{-1} x$ find $\frac{dy}{dx}$

☑ Soln. :

Given $y = (1 + x^2) \tan^{-1} x$

Differentiate both sides w.r. to x we get,

$$\frac{dy}{dx} = \underbrace{\frac{d}{dx}[(1 + x^2) \tan^{-1} x]}_{\frac{d}{dx}(u \cdot v)\text{ form}} \quad \ldots (1)$$

➔ Using Product rule of derivative

$$\ldots \left[\frac{d}{dx}(u.v.) = u\frac{d}{dx} v + v\frac{d}{dx} u \right]$$

$$\text{Equation (1)} \Rightarrow \frac{dy}{dx} = (1 + x^2) \underbrace{\frac{d}{dx} \tan^{-1} x}_{\frac{1}{1+x^2}} + \tan^{-1} x \underbrace{\frac{d}{dx} (1 + x^2)}_{(0 + 2x)}$$

➔ Using the standard derivatives :

$$\ldots \left[\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2} \text{ and } \frac{d}{dx}(1 + x^2) = 0 + 2x^{2-1} = 2x^1 = 2x \right]$$

$$\therefore \frac{dy}{dx} = \cancel{(1 + x^2)} \frac{1}{\cancel{1 + x^2}} + \tan^{-1} x (0 + 2x)$$

$$\therefore \frac{dy}{dx} = 1 + 2x \tan^{-1} x \checkmark$$ … simplification **…Ans.**

Ex. 10.1.6 W-12, 4 Marks.

Differentiate (sin x) cos x w.r. to x

☑ Soln. : Given, $y = (\sin x) \cos x$

Differentiate both sides w.r. to x we get

$$\frac{dy}{dx} = \underbrace{\frac{d}{dx}[\sin x \cdot \cos x]}_{\frac{d}{dx}(u \cdot v)\text{ form}} \quad \ldots (1)$$

➔ Using product rule of derivative :

$$\ldots \left[\frac{d}{dx}(u \cdot v) = u\frac{d}{dx} v + v\frac{d}{dx} u; \text{ Here, } u = \sin x \text{ and } v = \cos x \right]$$

$\therefore$ Equation (1) $\Rightarrow$

$$\frac{dy}{dx} = \sin x \underbrace{\frac{d}{dx} \cos x}_{-\sin x} + \cos x \underbrace{\frac{d}{dx} \sin x}_{\cos x}$$

➔ Using derivative of standard functions

$$\ldots \left[\frac{d}{dx}(\cos x) = -\sin x,\ \frac{d}{dx}(\sin x) = \cos x \right]$$

$$= \sin x (-\sin x) + \cos x (\cos x)$$

$$\therefore \frac{dy}{dx} = -\sin^2 x + \cos^2 x$$

$$\therefore \frac{dy}{dx} = \underbrace{\cos^2 x - \sin^2 x}_{\cos 2x}$$

$\because$ Standard formula $[\cos 2x = \cos^2 x - \sin^2 x]$

$$\frac{dy}{dx} = \cos 2x \checkmark$$ **…Ans.**

Ex. 10.5.7 S-11, 2 Marks.

Find $\frac{dy}{dx}$ if $y = \sec x \cdot \tan x$

☑ Soln. : Given, $y = \sec x \cdot \tan x$

Differentiate both sides w.r.to x we get

$$\frac{dy}{dx} = \underbrace{\frac{d}{dx}(\sec x \cdot \tan x)}_{\frac{d}{dx}(u \cdot v)\text{ form}} \quad \ldots (1)$$

➔ Using product rule of derivative

$$\ldots \left[\frac{d}{dx}(u \cdot v) = u\frac{d}{dx} v + v\frac{d}{dx} u; \text{ Here, } u = \sec x \text{ and } v = \tan x \right]$$

$$\therefore \text{Equation (1)} \Rightarrow \frac{dy}{dx} = \sec x \underbrace{\frac{d}{dx} \tan x}_{\sec^2 x} + \tan x \underbrace{\frac{d}{dx} \sec x}_{\sec x \tan x}$$

$$\therefore \frac{dy}{dx} = \underbrace{\sec x \cdot \sec^2 x}_{\sec^3 x} + \underbrace{\tan x (\sec x \tan x)}_{\sec x \cdot \tan^2 x}$$

$$\therefore \frac{dy}{dx} = \sec^3 x + \sec x \tan^2 x$$

$$= \sec x\ [\sec^2 x + \tan^2 x]$$ … Taking sec x common

$$\therefore \frac{dy}{dx} = \sec x [\sec^2 x + \tan^2 x] \checkmark$$ **…Ans.**

Ex. 10.1.8 **S-09, W-12, 2 Marks.**

Find $\frac{dy}{dx}$ if $y = e^x \cdot \tan x$

Soln. : Given, $y = e^x \tan x$

Differentiate both sides w.r.t. to x we get,

$$\frac{dy}{dx} = \underbrace{\frac{d}{dx}\left(e^x \tan x\right)}_{\frac{d}{dx}(u \cdot v)\ form} \quad \ldots (1)$$

Using product rule of derivative :

$$\ldots \left[\frac{d}{dx}(u \cdot v) = u\frac{d}{dx}v + v\frac{d}{dx}u\right]$$

… Here $u = e^x$ and $v = \tan x$

$$\therefore \text{ Equation (1)} \Rightarrow \frac{dy}{dx} = e^x \underbrace{\frac{d}{dx}\tan x}_{sec^2 x} + \tan x \underbrace{\frac{d}{dx}e^x}_{e^x}$$

$$\therefore \frac{dy}{dx} = e^x \sec^2 x + \tan x\, e^x$$

$$= e^x[\sec^2 x + \tan x] \quad \ldots \text{Taking } e^x \text{ common}$$

$$\Rightarrow \frac{dy}{dx} = e^x[\sec^2 x + \tan x] \checkmark \quad \ldots\text{Ans.}$$

Ex. 10.1.9 **S-15, 2 Marks.**

Find $\frac{dy}{dx}$ if $y = e^x \cdot \sin x$

Soln. :

Given, $y = e^x \sin x$

Differentiate both sides w.r.t. to x we get,

$$\frac{dy}{dx} = \underbrace{\frac{d}{dx}\left(e^x \sin x\right)}_{\frac{d}{dx}(u \cdot v)\ form} \quad \ldots (1)$$

Using Product rule of derivative

$$\ldots \left[\frac{d}{dx}(u \cdot v) = u\frac{d}{dx}v + v\frac{d}{dx}u\right]$$

… Here, $u = e^x$ and $v = \sin x$

$$\therefore \text{ Equation (1)} \Rightarrow \frac{dy}{dx} = e^x \underbrace{\frac{d}{dx}\sin x}_{\cos x} + \sin x \underbrace{\frac{d}{dx}e^x}_{e^x}$$

$$\therefore \frac{dy}{dx} = e^x \cos x + \sin x\, e^x$$

$$= e^x[\cos x + \sin x] \quad \ldots \text{Taking } e^x \text{ common}$$

$$\therefore \frac{dy}{dx} = e^x[\cos x + \sin x] \checkmark \quad \ldots\text{Ans.}$$

Ex. 10.1.10 **S-10, 2 Marks.**

Find $\frac{dy}{dx}$, if $y = (x + 1)(x + 2)$

Soln. : Given, $y = (x + 1)(x + 2)$

Differentiate both sides w.r.t. to x we get,

$$\frac{dy}{dx} = \underbrace{\frac{d}{dx}\left[(x+1)(x+2)\right]}_{\frac{d}{dx}(u \cdot v)\ form} \quad \ldots (1)$$

Using Product rule of derivative :

$$\ldots \left[\frac{d}{dx}(u \cdot v) = u\frac{d}{dx}v + v\frac{d}{dx}u\right]$$

$$\text{Equation (1)} \Rightarrow \frac{dy}{dx} = (x+1)\underbrace{\frac{d}{dx}(x+2)}_{1} + (x+2)\underbrace{\frac{d}{dx}(x+1)}_{1}$$

$$= (x+1)\cdot(1) + (x+2)(1)$$

$$\left[\begin{array}{l} \because \frac{d}{dx}(x+2) = \frac{d}{dx}x + \frac{d}{dx}2 = 1 + 0 = 1 \\ \text{and } \frac{d}{dx}(x+1) = \frac{d}{dx}x + \frac{d}{dx}1 = 1 + 0 = 1 \end{array}\right]$$

$$\frac{dy}{dx} = (x+1) + (x+2)$$

$$= \underbrace{x + 1 + x + 2}_{x + x + 1 + 2 \;=\; 2x + 3}$$

$$\frac{dy}{dx} = 2x + 3 \checkmark \quad \ldots\text{Ans.}$$

Ex. 10.5.11 **W-07, W-12, 2 Marks.**

If $\frac{dy}{dx}$ if $y = \frac{e^x + 1}{e^x - 1}$

Soln. : Given, $y = \frac{e^x + 1}{e^x - 1} \quad \ldots (1)$

$\therefore$ Differentiate both sides w.r.t. to x we get,

$$\therefore \text{Equation (1)} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x+1}{e^x-1}\right)\Big\} \quad \frac{d}{dx}\left(\frac{u}{v} form\right) \quad \ldots(2)$$

Using by quotient rule of derivative :

$$\ldots \left[\frac{dy}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}\right]$$

Here, $u = e^x + 1$ and $v = e^x - 1$

$\therefore$ Equation (2) becomes,

$$\frac{dy}{dx} = \frac{(e^x - 1)\frac{d}{dx}(e^x + 1) - (e^x + 1)\frac{d}{dx}(e^x - 1)}{(e^x - 1)^2}$$

$$\frac{dy}{dx} = \frac{(e^x - 1)\left(\frac{d}{dx}e^x + \frac{d}{dx}1\right) - (e^x + 1)\left(\frac{d}{dx}e^x - \frac{d}{dx}1\right)}{(e^x - 1)^2}$$

Using by derivative of standard function :

$$\ldots \left[\frac{d}{dx}e^x = e^x, \frac{d}{dx}1 = 0\right]$$

$$= \frac{(e^x-1)(e^x+0)-(e^x+1)(e^x-0)}{(e^x-1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{(e^x-1)(e^x)-(e^x+1)e^x}{(e^x-1)^2}$$

$$= \frac{e^x e^x - e^x - e^x e^x - e^x}{(e^x-1)^2}$$ …on simplification

$$\therefore \frac{dy}{dx} = \frac{-e^x-e^x}{(e^x-1)^2} \rightarrow -2e^x$$

$$\therefore \frac{dy}{dx} = \frac{-2e^x}{(e^x-1)^2}$$ ✓ …Ans.

Ex. 10.1.12 W-14, 4 Marks.

If $y = \frac{e^x+e^{-x}}{e^x-e^{-x}}$, find $\frac{dy}{dx}$

☑ **Soln. :** Given, $y = \frac{e^x+e^{-x}}{e^x-e^{-x}}$ …(1)

∴ Differentiate Equation (1) both sides w.r.to x we get,

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right) \quad \frac{d}{dx}\left(\frac{u}{v} form\right) \quad …(2)$$

➔ **Using Quotient rule of derivative**

$$…\left[\frac{dy}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}\right]$$

with $u = e^x + e^{-x}$ and $v = e^x - e^{-x}$

∴ Equation (2) ⇒

$$\frac{dy}{dx} = \frac{(e^x-e^{-x})\frac{d}{dx}(e^x+e^{-x}) - (e^x+e^{-x})\frac{d}{dx}(e^x-e^{-x})}{(e^x-e^{-x})^2}$$

$$= \frac{(e^x-e^{-x})\left(\frac{d}{dx}e^x + \frac{d}{dx}e^{-x}\right) - (e^x+e^{-x})\left(\frac{d}{dx}e^x - \frac{d}{dx}(e^{-x})\right)}{(e^x-e^{-x})^2}$$

➔ **Using by derivative of standard function**

$$…\left[\frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}e^{-x} = -e^{-x}\right]$$

$$= \frac{(e^x-e^{-x})(e^x-e^{-x}) - (e^x+e^{-x})(e^x-(-e^{-x}))}{(e^x-e^{-x})^2}$$

$$\therefore \frac{dy}{dx} = \frac{\overbrace{(e^x-e^{-x})(e^x-e^{-x})}^{(e^x-e^{-x})^2} - \overbrace{(e^x+e^{-x})(e^x+e^{-x})}^{(e^x+e^{-x})^2}}{(e^x-e^{-x})^2}$$

$$\therefore \frac{dy}{dx} = \frac{\overbrace{(e^x-e^{-x})^2}^{(a-b)^2} - \overbrace{(e^x+e^{-x})^2}^{(a+b)^2}}{(e^x-e^{-x})^2} \quad …(3)$$

$\because (a-b)^2 = a^2 - 2ab + b^2$

$$\therefore (e^x-e^{-x})^2 = (e^x)^2 - 2e^x \cdot e^{-x} + (e^{-x})^2 = e^{2x} - 2 + e^{-2x}$$

and $(a+b)^2 = a^2 + 2ab + b^2$

$$\therefore (e^x+e^{-x})^2 = (e^x)^2 + 2(e^x)\cdot(e^{-x}) + (e^{-x})^2$$

$$= e^{2x} + 2 + e^{-2x}$$

∴ Equation (3) becomes,

$$\frac{dy}{dx} = \frac{(e^{2x}-2+e^{-2x}) - (e^{2x}+2+e^{-2x})}{(e^x-e^{-x})^2}$$

$$= \frac{e^{2x}-2+e^{-2x}-e^{2x}-2-e^{-2x}}{(e^x-e^{-x})^2}$$ …on simplification

$$\therefore \frac{dy}{dx} = \frac{-2-2}{(e^x-e^{-x})^2}$$

$$\therefore \frac{dy}{dx} = \frac{-4}{(e^x-e^{-x})^2}$$ ✓ …Ans.

Ex. 10.1.13 S-10, S-12, 2 Marks.

Find $\frac{dy}{dx}$, if $y = \frac{\sin x}{1+\cos x}$

☑ **Soln. :**

Given, $y = \frac{\sin x}{1+\cos x}$

Differential both sides w.r.to x we get

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right) \begin{matrix}\rightarrow u \\ \rightarrow v\end{matrix} \quad …(1)$$

➔ **Using by Quotient rule of derivative :**

$$…\left[\frac{dy}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{(v)^2}\right]$$

∴ Equation (1)

$$\Rightarrow \frac{dy}{dx} = \frac{(1+\cos x)\frac{d}{dx}\sin x - \sin x\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{(1+\cos x)\frac{d}{dx}\sin x - \sin x\left(\frac{d}{dx}1 + \frac{d}{dx}\cos x\right)}{(1+\cos x)^2}$$

➔ **Using by derivative of standard function :**

$$…\left[\frac{d}{dx}\sin x = \cos x, \frac{d}{dx}\cos x = -\sin x\right]$$

$$\therefore \frac{dy}{dx} = \frac{(1+\cos x)(\cos x) - \sin x(0+(-\sin x))}{(1+\cos x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{(1+\cos x)(\cos x) - \sin x(0-\sin x)}{(1+\cos x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{\cos x + \overbrace{(\cos x\cdot\cos x) - (-\sin x \sin x)}^{\cos^2 x + \sin^2 x}}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\overbrace{\cos x + \cos^2 x + \sin^2 x}^{1}}{(1+\cos x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1+\cancel{\cos x}}{(1+\cos x)^{\cancel{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{\mathbf{1}}{\mathbf{(1+\cos x)}} \quad ✓ \qquad \text{...Ans..}$$

Ex. 10.1.14 **S-16, 2 Marks.**

Find $\frac{dy}{dx}$, if $y = \frac{\sin x}{1-\cos x}$

☑ **Soln. :** **Given :** $y = \frac{\sin x}{1-\cos x}$

Differentiate both sides w.r.t. x, it gives,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\sin x}{1-\cos x}\right]$$

➔ **Using Quotient rule of derivative :**

$$\ldots \left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\cdot\frac{du}{dx} - u\cdot\frac{dv}{dx}}{v^2} \quad \text{Here, } u = \sin x,\ v = (1-\cos x)\right]$$

$$\therefore \frac{dy}{dx} = \frac{(1-\cos x)\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(1-\cos x)}{(1-\cos x)^2}$$

➔ **Using derivatives of standard function**

$$\ldots \left[\frac{d}{dx}\sin x = \cos x \text{ and } \frac{d}{dx}\cos x = -\sin x\right]$$

$$\therefore \frac{du}{dx} = \frac{(1-\cos x)(\cos x) - \sin x\,(0+\sin x)}{(1-\cos x)^2}$$

$$\therefore \frac{du}{dx} = \frac{\cos x\,(1-\cos x) - \sin x\,(\sin x)}{(1-\cos x)^2}$$

$$= \frac{(\cos x)(1) \overbrace{- \cos x\cdot\cos x - \sin x\cdot\sin x}^{-\cos^2 x - \sin^2 x}}{(1-\cos x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{\cos x - \cos^2 x - \sin^2 x}{(1-\cos x)^2}$$

$$= \frac{\cos x - \left(\cos^2 x + \sin^2 x\right)}{(1-\cos x)^2} \qquad \text{...(by simplification)}$$

➔ **Using standard trigonometric formula :**

$$\ldots [\sin^2 x + \cos^2 x = 1]$$

$$= \frac{\cos x - 1}{(1-\cos x)^2}$$

$$= -\frac{\cancel{(\cos x - 1)}}{(\cos x - 1)^{\cancel{2}}}$$

$$\therefore \frac{\mathbf{dy}}{\mathbf{dx}} = -\frac{\mathbf{1}}{\mathbf{\cos x - 1}} = \frac{\mathbf{1}}{\mathbf{1-\cos x}} \quad ✓ \qquad \text{...Ans.}$$

Ex. 10.1.15 : Find derivative of $\frac{x^5 - \cos x}{\sin x}$ w.r. to x

☑ **Soln. :** Consider, $y = \frac{x^5 - \cos x}{\sin x}$

Differentiate both sides w.r. to x we get

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^5 - \cos x}{\sin x}\right) \qquad \ldots \frac{d}{dx}\left(\frac{u}{v}\right) \text{ form}$$

➔ **Using Quotient rule of derivative :**

$$\ldots \left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,\frac{d}{dx}u - u\,\frac{d}{dx}v}{v^2}\right]$$

$$\therefore \frac{dy}{dx} = \frac{\sin x\,\frac{d}{dx}(x^5 - \cos x) - (x^5 - \cos x)\,\frac{d}{dx}\sin x}{(\sin x)^2}$$

$$= \frac{\sin x\left[\frac{d}{dx}x^5 - \frac{d}{dx}\cos x\right] - (x^5 - \cos x)\,\frac{d}{dx}\sin x}{(\sin x)^2}$$

➔ **Using derivative of standard function**

$$\ldots \left[\frac{d}{dx}x^5 = 5x^{5-1} = 5x^4;\ \frac{d}{dx}\cos x = -\sin x,\ \frac{d}{dx}\sin x = \cos x\right]$$

$$= \frac{\sin x\left[5x^{5-1} - (-\sin x)\right] - (x^5 - \cos x)\ (\cos x)}{(\sin x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{\sin x\left(5x^4 + \sin x\right) - (x^5 - \cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{5x^4\ \sin x + \sin^2 x - x^5\cos x + \cos^2 x}{\sin^2 x}$$

$$= \frac{5x^4\ \sin x - x^5\cos x + \overbrace{\left(\sin^2 x + \cos^2 x\right)}^{1}}{\sin^2 x}$$

$$\therefore \frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\mathbf{5x^4\ \sin x - x^5\cos x + 1}}{\mathbf{\sin^2 x}} \quad ✓ \qquad \text{...Ans.}$$

Ex. 10.1.16 **S-11, 2 Marks.**

Find $\frac{dy}{dx}$ if $y = \frac{x+1}{x-1}$

☑ **Soln. :** Given, $y = \frac{x+1}{x-1}$

Differentiate w.r. to x we get,

$$\frac{dy}{dx} = \underbrace{\frac{d}{dx}\left(\frac{x+1}{x-1}\right)}_{\frac{u}{v}\ form} \qquad \ldots (1)$$

➔ **Using Quotient rule of derivative**

$$\ldots \left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,\frac{d}{dx}u - u\,\frac{d}{dx}v}{v^2}\right]$$

Equation (1) $\Rightarrow \dfrac{dy}{dx} = \dfrac{(x-1)\dfrac{d}{dx}(x+1) - (x+1)\dfrac{d}{dx}(x-1)}{(x-1)^2}$

$$= \frac{(x-1)\left(\overbrace{\frac{d}{dx}x}^{1} + \overbrace{\frac{d}{dx}1}^{0}\right) - (x+1)\left(\overbrace{\frac{d}{dx}x}^{1} - \overbrace{\frac{d}{dx}1}^{0}\right)}{(x-1)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x-1)\ (1+0) - (x+1)\ (1-0)}{(x-1)^2}$$

$$\left[\because \frac{d}{dx}(x) = 1 \text{ and } \frac{d}{dx}(1) = 0\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x-1)\ (1) - (x+1)\ (1)}{(x-1)^2}$$

$$= \frac{\not{x} - 1 - \not{x} - 1}{(x-1)^2} \quad \ldots \text{ on simplification}$$

$$= \frac{-1-1}{(x-1)^2}$$

$$\therefore \quad \mathbf{\frac{dy}{dx} = \frac{-2}{(x-1)^2}} \ ✓ \qquad \ldots\textbf{Ans.}$$

Ex. 10.1.17 **S-2011, 2 Marks.**

Find $\dfrac{dy}{dx}$ if $y = \sin(2x+1)$

☑ **Soln. :** Given, $y = \sin(2x+1)$

Differentiate w.r. to x on both sides,

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[\sin(2x+1)] \qquad \ldots (1)$$

We have,

$$\left[\begin{array}{l} \dfrac{d}{dx}\sin(ax+b) = \cos(ax+b)\cdot a \\ \Rightarrow \dfrac{d}{dx}\sin(2x+1) = \cos(2x+1)\cdot 2 \end{array}\right]$$

$$\therefore \quad \frac{dy}{dx} = \cos(2x+1)\cdot 2$$

$$\therefore \quad \mathbf{\frac{dy}{dx} = 2\cos(2x+1)} \ ✓ \qquad \ldots\textbf{Ans.}$$

Ex. 10.1.18 **S-2009, 2 Marks.**

Find $\dfrac{dy}{dx}$, if $y = \cos^2 x$

☑ **Soln. :** Given, $y = \cos^2 x$

Differentiate w.r. to x we get

$$\therefore \quad \frac{dy}{dx} = \underbrace{\frac{d}{dx}(\cos x)^2}_{\frac{d}{dx}(\cos x)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(\cos x)^2$$

By derivative of composite function,

$$\therefore \quad \frac{dy}{dx} = 2(\cos x)^{2-1}\underbrace{\frac{d}{dx}(\cos x)}_{-\sin x}$$

$$= 2(\cos x)^1\ (-\sin x)$$

$$= \underbrace{-2\cos x \sin x}_{\sin 2x}$$

$$= -\sin 2x$$

$$\therefore \quad \mathbf{\frac{dy}{dx} = -\sin 2x} \ ✓ \qquad \ldots\textbf{Ans.}$$

Ex. 10.1.19 **S-2010, 2 Marks.**

Find $\dfrac{dy}{dx}$, if $y = \sin^3 x$

☑ **Soln. :** Given, $y = \sin^3 x \Rightarrow y = (\sin x)^3$

Differentiate w.r. to x on both sides we get

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(\sin x)^3$$

By derivative of composite function

$$\therefore \quad \frac{dy}{dx} = 3(\sin x)^{3-1}\underbrace{\frac{d}{dx}(\sin x)}_{\cos x}$$

$$= 3\underbrace{(\sin x)^2}_{\sin^2 x}\cos x$$

$$\therefore \quad \mathbf{\frac{dy}{dx} = 3\sin^2 x \cos x} \ ✓ \qquad \ldots\textbf{Ans.}$$

Ex. 10.1.20 **W-2009, 2 Marks.**

Find $\dfrac{dy}{dx}$ if $y = 2^x + \cos 3x$

☑ **Soln. :**

$$\text{Given,} \quad y = \underbrace{2^x}_{a^x} + \underbrace{\cos 3x}_{\cos ax} \qquad \ldots (1)$$

Differentiate w.r. to x on both sides we get

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[2^x + \cos 3x]$$

$$\therefore \quad \frac{dy}{dx} = \underbrace{\frac{d}{dx}2^x}_{2^x \log 2} + \underbrace{\frac{d}{dx}\cos 3x}_{(-\sin 3x)\frac{d}{dx}3x}$$

$$\left\{\because \frac{d}{dx}a^x = a^x \log a, \text{ Here } a = 2 \text{ and } \frac{d}{dx}(\cos ax) = (-\sin ax)\cdot\frac{d}{dx}(ax), \text{ Here } a = 3\right\}$$

$$\therefore \quad \frac{dy}{dx} = 2^x \log 2 + (-\sin 3x)\underbrace{\frac{d}{dx}(3x)}_{3} \qquad \because \frac{d}{dx}(ax) = a$$

$$\therefore \quad \frac{dy}{dx} = 2^x \log 2 - \underbrace{\sin 3x \cdot 3}_{-3\sin 3x}$$

$$\therefore \quad \mathbf{\frac{dy}{dx} = 2^x \log 2 - 3\sin 3x} \ ✓ \qquad \ldots\textbf{Ans.}$$

Ex. 10.1.21 W-2012, 2 Marks.

Find $\frac{dy}{dx}$ if $y = \tan(4-3x)$

☑ **Soln. :** Given, $y = \tan(4-3x)$

Differentiate both sides w.r. to x we get

$$\frac{d}{dx} y = \frac{d}{dx}[\tan(4-3x)] \quad \ldots(1)$$

$\because$ We have, $\frac{d}{dx}\tan[f(x)] = \sec^2 f(x)\left[\frac{d}{dx} f(x)\right]$

$\therefore$ Equation (1) $\Rightarrow \frac{d}{dx} y = \sec^2(4-3x)\underbrace{\frac{d}{dx}(4-3x)}_{\frac{d}{dx}4 - \frac{d}{dx}3x}$

$\because$ Here $f(x) = 4-3x$

$$\therefore \frac{dy}{dx} = \sec^2(4-3x)\left[\underbrace{\frac{d}{dx}4}_{0} - \underbrace{\frac{d}{dx}3x}_{3}\right]$$

➔ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\right]$$

$$\therefore \frac{dy}{dx} = \sec^2(4-3x)(0-3) = \sec^2(4-3x)(-3)$$

$$= -3\sec^2(4-3x) \quad \ldots \text{Rearranging terms}$$

$$\therefore \mathbf{\frac{dy}{dx} = -3\sec^2(4-3x)} \checkmark \quad \ldots\text{Ans.}$$

Ex. 10.1.22 W-2010, 2 Marks.

Find $\frac{dy}{dx}$, if $y = e^{3x}\sin 2x$

☑ **Soln. :** Given : $y = e^{3x}\sin 2x$

Differentiate both sides w.r.t. x we get

$$\therefore \frac{dy}{dx} = \underbrace{\frac{d}{dx}[e^{3x}\sin 2x]}_{\frac{d}{dx}(u\cdot v)\ form} \quad \ldots.(1)$$

➔ **Using product rule** $\ldots\left[\frac{d}{dx}(u\cdot v) = u\frac{d}{dx}v + v\frac{d}{dx}u\right]$

$\therefore$ Equation (1) becomes,

$$\frac{dy}{dx} = e^{3x}\underbrace{\frac{d}{dx}(\sin 2x)}_{\cos 2x\frac{d}{dx}(2x)} + \sin 2x\underbrace{\frac{d}{dx}(e^{3x})}_{e^{3x}\frac{d}{dx}(3x)}$$

$$\therefore \frac{dy}{dx} = e^{3x}\left[\cos 2x\underbrace{\frac{d}{dx}(2x)}_{2}\right] + \sin 2x\left[e^{3x}\underbrace{\frac{d}{dx}(3x)}_{3}\right]$$

… By derivative of composite function

$$\therefore \frac{dy}{dx} = e^{3x}(\cos 2x(2)) + \sin 2x[e^{3x}(3)]$$

$$= 2e^{3x}\cos 2x + 3e^{3x}\sin 2x \quad \ldots \text{Rearranging}$$

$$= e^{3x}[2\cos 2x + 3\sin 2x] \quad \ldots \text{taking common } e^{3x}$$

$$\therefore \mathbf{\frac{dy}{dx} = e^{3x}[2\cos 2x + 3\sin 2x]} \checkmark \quad \ldots\text{Ans.}$$

Ex. 10.1.23 W-2016, 2 Marks.

If $y = e^{7x}\cos 7x$, find $\frac{dy}{dx}$

☑ **Soln. :** $y = e^{7x}\cos 7x$;

Differentiate both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{7x}\underbrace{\frac{d}{dx}\cos 7x}_{-\sin 7x.7} + \cos 7x\underbrace{\frac{d}{dx}e^{7x}}_{e^{7x}\cdot 7}$$

➔ **Using Derivative of standard function :**

$$\ldots\left[\frac{d}{dx}\cos ax = (-\sin ax)\cdot(a) \text{ and } \frac{d}{dx}e^{ax} = e^{ax}\cdot a\right]$$

$$\therefore \frac{dy}{dx} = e^{7x}(-\sin 7x)\cdot 7 + \cos 7x\, e^{7x}\cdot 7$$

Taking common $(7e^{7x})$,

$$\therefore \mathbf{\frac{dy}{dx} = 7e^{7x}(-\sin 7x + \cos 7x)} \checkmark \quad \ldots\text{Ans.}$$

Ex. 10.1.24 W-2015, 2 Marks.

If $y = e^{4x}\cos 3x$ find $\frac{dy}{dx}$

☑ **Soln. : Given :** $y = e^{4x}\cos 3x$

Differentiate w.r.t x, it gives

$$\frac{dy}{dx} = \underbrace{\frac{d}{dx}[e^{4x}\cos 3x]}_{\frac{d}{dx}(u\cdot v)\ form}$$

➔ **Using product rule of derivative**

$$\ldots\left[\frac{d}{dx}(u\cdot v) = u\cdot\frac{dv}{dx} + v\frac{du}{dx}\right]$$

Here, $u = e^{4x}$ and $v = \cos 3x$

$$= e^{4x}\underbrace{\frac{d}{dx}[\cos 3x]}_{-\sin 3x} + \cos 3x\underbrace{\frac{d}{dx}[e^{4x}]}_{4e^{4x}} \ldots\text{(by product rule)}$$

$$= e^{4x}\left[-\sin 3x\underbrace{\frac{d}{dx}(3x)}_{3}\right] + \cos 3x\, e^{4x}\underbrace{\frac{d}{dx}(4x)}_{4}$$

…(by derivative of composite function)

$$= e^{4x}[-\sin 3x(3)] + \cos 4x\, e^{4x}(4)$$

$$= -3e^{4x}\sin 3x + 4e^{4x}\cos 4x$$

$$\mathbf{\frac{dy}{dx} = e^{4x}(-3\sin 3x + \cos 3x)} \checkmark \quad \ldots\text{Ans.}$$

Ex. 10.1.25 W-2015, 2 Marks.

If $y = \log [\sin (4x - 3)]$ find $\frac{dy}{dx}$.

Soln. :

Given : $y = \log \sin (4x - 3)$

Differentiate both sides w.r.t. x, it gives,

$$\frac{dy}{dx} = \frac{d}{dx}[\log \sin (4x - 3)]$$

➔ **Using derivative of composite function**

$$\ldots\left[\frac{d}{dx}\log f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)\right]$$

Here, $f(x) = \sin (4x - 3)$

$$= \frac{1}{\sin (4x-3)} \cdot \frac{d}{dx}[\sin (4x - 3)] \quad \ldots (1)$$

Here, $f(x) = \sin (4x - 3)$

Now we have, $\frac{d}{dx}\sin (4x - 3) = \cos (4x - 3)\frac{d}{dx}(4x - 3)$

… [by composite function derivative]

∴ Equation (1) becomes

$$= \underbrace{\frac{1}{\sin (4x-3)}\cos(4x - 3)}_{\cot (4x-3)}\ \underbrace{\frac{d}{dx}(4x - 3)}_{\underbrace{\frac{d}{dx}(4x) - \frac{d}{dx}(3)}_{4-0}}$$

…[by standard derivatives]

$$\therefore \frac{dy}{dx} = 4 \cot (4x - 3) \checkmark \quad \text{...Ans.}$$

Ex. 10.1.26 S-2010, S-2016, 2 Marks.

Find $\frac{dy}{dx}$, if $y = \log (\sec x + \tan x)$

Soln. :

Given, $y = \log (\sec x + \tan x)$

Differentiate both sides w.r. to x, we get

$$\therefore \frac{dy}{dx} = \frac{d}{dx}[\log (\sec x + \tan x)]$$

$$= \frac{1}{(\sec x + \tan x)}\frac{d}{dx}(\sec x + \tan x)$$

∵ By derivative of composite function

$$\therefore \frac{dy}{dx}\log [f(x)] = \frac{1}{f(x)}\frac{d}{dx}[f(x)]$$

$$\therefore \frac{dy}{dx} = \frac{1}{(\sec x + \tan x)}\left[\underbrace{\frac{d}{dx}\sec x}_{\sec x \tan x} + \underbrace{\frac{d}{dx}\tan x}_{\sec^2 x}\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{(\sec x + \tan x)}[\sec x \tan x + \sec^2 x]$$

$$= \frac{\sec x}{\sec x + \tan x}[\tan x + \sec x]$$

… Taking sec x common

$$\therefore \frac{dy}{dx} = \frac{\sec x}{\cancel{\sec x + \tan x}}\cancel{(\sec x + \tan x)}$$

….Rearranging Terms

$$\therefore \frac{dy}{dx} = \sec x \checkmark \quad \text{...Ans.}$$

Ex. 10.1.27 W-2007, W-2010, W-2012, 2 Marks.

Find $\frac{dy}{dx}$, if $y = \log (x^2 + 2x + 5)$

Soln. :

Given $y = \log (x^2 + 2x + 5)$

Differentiate both sides w.r. to x we get

$$\therefore \frac{dy}{dx} = \frac{d}{dx}[\log (x^2 + 2x + 5)]$$

➔ **Using derivative of composite formula**

$$\ldots\left[\frac{d}{dx}\log f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)\right]$$

$$= \frac{1}{x^2 + 2x + 5}\frac{d}{dx}(x^2 + 2x + 5)$$

➔ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\right]$$

$$\frac{dy}{dx} = \frac{1}{x^2 + 2x + 5}\left[\underbrace{\frac{d}{dx}x^2}_{2x} + \underbrace{\frac{d}{dx}2x}_{2} + \underbrace{\frac{d}{dx}5}_{0}\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{x^2 + 2x + 5}[2x + 2 + 0] \quad \text{….By derivative result}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x^2 + 2x + 5}[2x + 2]$$

$$= \frac{2(x + 1)}{x^2 + 2x + 5} \quad \text{…Taking 2 common}$$

$$\therefore \frac{dy}{dx} = \frac{2(x + 1)}{x^2 + 2x + 5} \checkmark \quad \text{...Ans.}$$

Ex. 10.1.28 W-2014, 2 Marks.

Find $\frac{dy}{dx}$, if $y = \log (x^2 + 2x)$

Soln. :

Given, $y = \log (x^2 + 2x)$

Differentiate both sides w.r.t. x we get

$$\frac{dy}{dx} = \frac{d}{dx}[\log (x^2 + 2x)]$$

$$= \frac{1}{x^2 + 2x}\frac{d}{dx}(x^2 + 2x)$$

➔ **Using derivative of composite function**

$$\ldots\left[\frac{d}{dx}\log f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x)\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{x^2+2x}\left[\underbrace{\frac{d}{dx}(x^2)}_{2x} + \underbrace{\frac{d}{dx}(2x)}_{2}\right]$$

$$\because \frac{d}{dx} = [f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

$$\therefore \frac{dy}{dx} = \frac{1}{x^2+2x}[2x+2]$$

$$= \frac{2(x+1)}{x(x+2)} \quad \text{… on simplification}$$

$$\mathbf{\frac{dy}{dx} = \frac{2(x+1)}{x(x+2)}} \checkmark \quad \text{….Ans.}$$

Ex. 10.1.29 W-2009, 2 Marks.

Differentiate $\sin^{-1}(\cos x)$ w.r.t x

☑ **Soln. :**

Consider, $y = \sin^{-1}(\cos x)$ …. (1)

∵By transformation formula, we have

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

∴Equation (1) becomes $y = \sin^{-1}\left[\sin\underbrace{\left(\frac{\pi}{2}-x\right)}_{\theta}\right]$

$$\therefore y = \underbrace{\frac{\pi}{2} - x}_{\theta} \qquad \because \sin^{-1}(\sin\theta) = \theta$$

Now differentiate both sides w.r.t. x we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\pi}{2} - x\right]$$

➔ **Using linearity property of derivative**

$$\text{…} \left[\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\right]$$

$$= \underbrace{\frac{d}{dx}\left(\frac{\pi}{2}\right)}_{0} - \underbrace{\frac{d}{dx}(x)}_{1}$$

$$\frac{dy}{dx} = \underbrace{0-1}_{-1}$$

$$\mathbf{\frac{dy}{dx} = -1} \checkmark \quad \text{…Ans.}$$

Ex. 10.1.30 S-2017, 4 Marks.

Differentiate w.r.t. x $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

☑ **Soln. :**

Consider, $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ …(1)

Put $x = \tan\theta$

$\Rightarrow \theta = \tan^{-1}x$ … (2)

∴ Equation (1) becomes

$$y = \tan^{-1}\left[\frac{2\tan\theta}{1-(\tan\theta)^2}\right]$$

$$= \tan^{-1}\underbrace{\left[\frac{2\tan\theta}{1-\tan^2\theta}\right]}_{\sin 2\theta} \qquad \because (\tan\theta)^2 = \tan^2\theta$$

$$\therefore y = \underbrace{\tan^{-1}(\sin 2\theta)}_{2\theta} \qquad \left[\because \text{By formula } \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}\right]$$

$$\therefore y = 2\theta \qquad \because \sin^{-1}(\sin\theta) = \theta$$

$$y = 2\tan^{-1}x \qquad \text{….By Equation (2)}$$

Now differentiate both sides w.r.t. x we get

$$\frac{dy}{dx} = \frac{d}{dx}[2\tan^{-1}x]$$

➔ **Using derivative property** $\text{…}\left[\frac{d}{dx}kf(x) = k\frac{d}{dx}f(x)\right]$

$$= 2\underbrace{\frac{d}{dx}\tan^{-1}x}_{\frac{1}{1+x^2}}$$

$$\therefore \frac{dy}{dx} = 2\frac{1}{1+x^2} \quad \text{… by derivative result}$$

$$\therefore \mathbf{\frac{dy}{dx} = \frac{2}{1+x^2}} \checkmark \quad \text{…Ans.}$$

Ex. 10.1.31 W-2014, 4 Marks.

Differentiate $\tan^{-1}\left(\frac{5x}{1-6x^2}\right)$ w.r.t x

☑ **Soln. :**

Consider, $y = \tan^{-1}\underbrace{\left(\frac{5x}{1-6x^2}\right)}_{\frac{3x+2x}{1-(3x)(2x)}}$ … (1)

$\because 5x = 3x + 2x$

and $6x^2 = (3 \times 2)(x \cdot x) = (3x)(2x)$

∴ Equation (1) becomes

$$y = \tan^{-1}\left[\frac{3x+2x}{1-(3x)(2x)}\right]$$

➔ **Using trigonometric formula**

$$\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}x + \tan^{-1}y\right]$$

Here x = 3x, y = 2x

$$\therefore y = \tan^{-1}(3x) + \tan^{-1}(2x)$$

Now differentiate both side w.r.t. x, we get,

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(3x) + \tan^{-1}(2x)]$$

➔ **Using linearity property of derivitive**

$$\ldots\left[\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\right]$$

$$= \frac{d}{dx}\tan^{-1}(3x) + \frac{d}{dx}\tan^{-1}(2x)$$

➔ **Using derivative of composite function**

$$\ldots\left[\frac{d}{dx}\tan^{-1}[f(x)] = \frac{1}{1+[f(x)]^2}\frac{d}{dx}f(x)\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+(3x)^2}\underbrace{\frac{d}{dx}(3x)}_{3} + \frac{1}{1+(2x)^2}\underbrace{\frac{d}{dx}(2x)}_{2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\underbrace{1+[3x]^2}_{9x^2}}\cdot(3) + \frac{1}{\underbrace{1+[2x]^2}_{4x^2}}\cdot(2)$$

$$\therefore \frac{dy}{dx} = \frac{3}{1+9x^2} + \frac{2}{1+4x^2} \checkmark$$ …**Ans.**

Ex. 10.1.32 **.W-2008, 4 Marks, W-2011, 2 Marks.**

Find $\frac{dy}{dx}$, if $y = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$

☑ **Soln. :**

Given : $y = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$ … (1)

Now, we have by half angle formula

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

and $1+\cos x = 2\cos^2\frac{x}{2}$

∴ Equation (1) becomes,

$$\therefore y = \tan^{-1}\left[\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right]$$

$$\therefore y = \tan^{-1}\underbrace{\left[\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right]}_{\tan\frac{x}{2}}$$

$$y = \tan^{-1}\underbrace{\left[\tan\frac{x}{2}\right]}_{\frac{x}{2}} \quad \ldots\left[\frac{\sin x}{\cos x} = \tan x\right]$$

$$\therefore y = \frac{x}{2} \quad [\because \tan^{-1}(\tan\theta) = \theta]$$

Differentiate both sides w.r.t x we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right)$$

➔ **Using derivative property** $\ldots\left[\frac{d}{dx}kf(x) = k\frac{d}{dx}f(x)\right]$

$$= \frac{1}{2}\underbrace{\frac{d}{dx}(x)}_{1}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}(1) = \frac{1}{2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \checkmark$$ …**Ans.**

Ex. 10.1.33 : Find $\frac{dy}{dx}$ if $y = \tan^{-1}\left(\frac{2x}{1+8x^2}\right)$

☑ **Soln. :**

Given : $y = \tan^{-1}\underbrace{\left(\frac{2x}{1+8x^2}\right)}_{\frac{4x-2x}{1+(4x)(2x)}}$ … (1)

$\because$ 2x = 4x – 2x and $8x^2 = (4\times 2)(x\cdot x) = 4x(2x)$

➔ **Using trigonometric formula**

$$\ldots\left[\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y\right]$$

Equation (1) becomes,

$$y = \tan^{-1}\left[\frac{4x-2x}{1+(4x)(2x)}\right] = \tan^{-1}(4x) - \tan^{-1}(2x)$$

$$y = \tan^{-1}(4x) - \tan^{-1}(2x)$$

Now differentiate both sides w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(4x) - \tan^{-1}(2x)]$$

➔ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\right]$$

$$= \frac{d}{dx}\tan^{-1}(4x) - \frac{d}{dx}\tan^{-1}(2x)$$

➔ **Using formula :** $\ldots\left[\frac{d}{dx}\tan^{-1}[f(x)] = \frac{1}{1+[f(x)]^2}\frac{d}{dx}f(x)\right]$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\frac{1}{1+(4x)^2}\underbrace{\frac{d}{dx}(4x)}_{4} - \frac{1}{1+(2x)^2}\underbrace{\frac{d}{dx}(2x)}_{2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+(4x)^2}4 - \frac{1}{1+(2x)^2} \quad \ldots (2)$$

$$\frac{dy}{dx} = \frac{4}{\underbrace{1+(4x)^2}_{16x^2}} - \frac{2}{\underbrace{1+(2x)^2}_{4x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{4}{1+16x^2} - \frac{2}{1+4x^2} \checkmark \qquad \ldots\textbf{Ans.}$$

Ex. 10.1.34 **S-2008, 4 Marks.**

Differentiate w.r.t x $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Soln. :

Consider, $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ … (1)

Put $x = \sin\theta$

$\therefore \theta = \sin^{-1}x$ … (2)

$\therefore$ Equation (1) becomes

$$y = \tan^{-1}\left(\frac{\sin\theta}{\underbrace{\sqrt{1-\sin\theta^2}}_{\cos^2\theta}}\right)$$

$$y = \tan^{-1}\left(\frac{\sin\theta}{\underbrace{\sqrt{\cos^2\theta}}_{\cos\theta}}\right) \qquad \because 1-\sin^2\theta = \cos^2\theta$$

$\therefore y = \tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right]$ $\qquad \because \sqrt{x^2} = x$

$\therefore y = \tan^{-1}(\tan\theta)$

$\therefore y = \theta$ $\qquad \because \tan^{-1}(\tan\theta) = \theta$

$\therefore y = \sin^{-1}x$ … By Equation (2)

Differentiate both sides wr.t x we get,

$$\frac{dy}{dx} = \underbrace{\frac{d}{dx}[\sin^{-1}x]}_{\frac{1}{\sqrt{1-x^2}}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \checkmark \qquad \ldots\textbf{Ans.}$$

Ex. 10.1.35 **S-2014, 4 Marks.**

If $y = \sin^{-1}(3x-4x^3)$, find $\frac{dy}{dx}$

Soln. :

Given, $y = \sin^{-1}(3x-4x^3)$ … (1)

Put, $x = \sin\theta$

$\Rightarrow \theta = \sin^{-1}x$ … (2)

Now putting value of x in Equation (1), we get

$$y = \sin^{-1}(3\sin\theta - 4\underbrace{(\sin\theta)^3}_{\sin^3\theta})$$

$$= \underbrace{\sin^{-1}[3\sin\theta - 4\sin^3\theta]}_{\sin 3\theta}$$

$$\therefore y = \underbrace{\sin^{-1}(\sin 3\theta)}_{3\theta}$$ $\because$ By formula $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\therefore y = 3\theta$ $\qquad \because \sin^{-1}(\sin 3\theta) = \theta$

$\therefore y = 3\sin^{-1}x$ … by Equation (2)

Now differentiate both sides w.r.t x we get,

$$\therefore \frac{dy}{dx} = \underbrace{\frac{d}{dx}(3\sin^{-1}x)}_{3\frac{d}{dx}(\sin^{-1}x)}$$

➔ **Using derivative property** …$\left[\frac{d}{dx}kf(x) = k\frac{d}{dx}f(x)\right]$

$$= 3\underbrace{\frac{d}{dx}(\sin^{-1}x)}_{\frac{1}{\sqrt{1-x^2}}} \qquad \ldots\text{by formula}$$

$$\therefore \frac{dy}{dx} = 3\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}} \checkmark \qquad \ldots\textbf{Ans.}$$

Ex. 10.1.36 **W-2008, 4 Marks, W-2011, 2 Marks.**

Find $\frac{dy}{dx}$ if $x^2 + y^2 = 25$

Soln. :

Given : $x^2 + y^2 = 25$

Differentiate both sides w.r.t x we get,

$$\frac{d}{dx}[x^2+y^2] = \frac{d}{dx}(25)$$

➔ **Using linearity property of derivative**

…$\left[\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\right]$

$$\underbrace{\frac{d}{dx}x^2}_{2x} + \underbrace{\frac{d}{dx}y^2}_{2y\frac{dy}{dx}} = 0$$

$\therefore 2x + 2y\frac{dy}{dx} = 0$ … [by derivative of standard function]

$2\left[x + y\frac{dy}{dx}\right] = 0$ … Taking 2 common

$\therefore x + y\frac{dy}{dx} = 0$ … As $mx = 0 \Rightarrow x = 0$ Here $m = 2$

$\therefore y\frac{dy}{dx} = -x$ … Shifting x towards right

$\therefore \frac{dy}{dx} = \frac{-x}{y}$ ✓ **… divide by y** **…Ans.**

Ex. 10.1.37 **S-2008, S-2013,S-2015, 4 Marks.**

If $(x^2 + y^2) = xy$, find $\frac{dy}{dx}$

Soln. :

Given : $(x^2 + y^2) = xy$

Differentiate w.r.t.x we get

$$\therefore \underbrace{\frac{d}{dx}(x^2+y^2)}_{\frac{d}{dx}[f(x)+g(x)]} = \underbrace{\frac{d}{dx}(xy)}_{\frac{d}{dx}[u\cdot v]} \quad \ldots (1)$$

definition of product rule

➔ **Using linearity property of derivative on L.H.S.**

$$\ldots\left[\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\right]$$

➔ **Using product rule of derivative on R.H.S.**

$$\ldots\left[\frac{d}{dx}[u\cdot v] = u\frac{d}{dx}v + v\frac{d}{dx}u\right]$$

∴ Equation (1) becomes,

$$\underbrace{\frac{d}{dx}x^2}_{2x} + \underbrace{\frac{d}{dx}y^2}_{2y\frac{dy}{dx}} = x\underbrace{\frac{d}{dx}y}_{\frac{dy}{dx}} + y\underbrace{\frac{d}{dx}x}_{1}$$

$$\therefore 2x + 2y\frac{dy}{dx} = x\frac{dy}{dx} + y(1)$$

... [by derivative of standard function]

$$\therefore \underbrace{2y\frac{dy}{dx} - x\frac{dy}{dx}}_{(2y-x)\frac{dy}{dx}} = y - 2x \quad \ldots \text{Rearrange terms}$$

$$\therefore (2y-x)\frac{dy}{dx} = (y-2x)$$

$$\therefore \frac{dy}{dx} = \frac{y-2x}{2y-x} \checkmark \quad \text{....Ans.}$$

Divide by $(2y - x)$ or As $ax = y \Rightarrow a = \frac{y}{x}$

Ex. 10.1.38 **.W-2012, 2 Marks.**

If $x^2 + y^2 + xy - y = 0$, find $\frac{dy}{dx}$ at (1,2)

Soln. : Given : $x^2 + y^2 + xy - y = 0$

Differentiate both sides w.r.t. x we get

$$\frac{d}{dx}[x^2 + y^2 + xy - y] = 0$$

➔ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\right]$$

$$\underbrace{\frac{d}{dx}x^2}_{2x} + \underbrace{\frac{d}{dx}y^2}_{2y\frac{dy}{dx}} + \frac{d}{dx}(xy) - \frac{dy}{dx} = 0$$

$$\therefore 2x + 2y\frac{dy}{dx} + \underbrace{\left[x\frac{d}{dx}y + y\frac{d}{dx}x\right]}_{\textit{By product rule}} - \frac{dy}{dx} = 0$$

$$\therefore 2x + 2y\frac{dy}{dx} + x\frac{dy}{dx} + y(1) - \frac{dy}{dx} = 0 \quad \text{...On simplification}$$

$$\therefore 2y\frac{dy}{dx} + x\frac{dy}{dx} - \frac{dy}{dx} = -2x - y(1)$$

[Collecting terms of $\frac{dy}{dx}$ on LHS and remaining on RHS]

$$\therefore (2y + x - 1)\frac{dy}{dx} = -(2x+y) \quad \text{....on simplification}$$

$$\therefore \frac{dy}{dx} = \frac{-(2x+y)}{(2y+x-1)} \quad \left[\because f(x)\frac{dy}{dx} = f(y) \Rightarrow \frac{dy}{dx} = \frac{f(y)}{f(x)}\right]$$

Now put x = 1 and y = 2 as we have to find $\frac{dy}{dx}$ at (1, 2)

$$\therefore \frac{dy}{dx} = \frac{-[2(1)+2]}{[\underbrace{2(2)}_{4} + \not{1} - \not{1}]}$$

$$= \frac{-[2+2]}{[4+0]} = \underbrace{\frac{-4}{4}}_{-1}$$

$$\therefore \frac{dy}{dx} = -1\checkmark \quad \text{...Ans.}$$

Ex. 10.1.39

.W-09,2 Marks, S-11,W-11,W-08, W-12, 4 Marks.

If $x^3 + y^3 = 3\,axy$, find $\frac{dy}{dx}$ at point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Soln. : Given : $x^3 + y^3 = 3\,axy$

Differentiate both sides w.r.t x we get

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(3\,axy)$$

➔ **Using linearity property of derivative on L.H.S.**

$$\ldots\left[\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\right]$$

➔ **Using derivative property on R.H.S.**

$$\ldots\left[\frac{d}{dx}kf(x) = k\frac{d}{dx}f(x)\right]$$

Here, k = 3a

$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = 3a\frac{d}{dx}(xy)$$

$$\underbrace{\frac{d}{dx}x^3}_{3x^2} + \underbrace{\frac{d}{dx}y^3}_{3y^2\frac{dy}{dx}} = 3a\frac{d}{dx}(xy) \quad \ldots(1)$$

Now we have, $\frac{d}{dx} x^n = n x^{n-1}$ $\therefore$ $\frac{d}{dx} x^3 = 3x^{3-1} = 3x^2$

Similarly, $\frac{d}{dx} y^3 = 3y^2 \frac{dy}{dx}$

Also by derivative of product rule,

$$\frac{d}{dx}(xy) = x\frac{d}{dx}y + y\frac{d}{dx}x = x\frac{dy}{dx} + y\frac{dx}{dx}$$

$\therefore$ Equation (1) becomes,

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\underbrace{\frac{dx}{dx}}_{1}\right]$$

$$\Rightarrow \quad 3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$$

$$\Rightarrow \quad 3x^2 + 3y^2\frac{dy}{dx} = 3ax\frac{dy}{dx} + 3ay$$

$$\Rightarrow 3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay - 3x^2 \quad \text{....Rearranging terms}$$

$$\Rightarrow (3y^2 - 3ax)\frac{dy}{dx} = 3ay - 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3ay - 3x^2}{3y^2 - 3ax} \quad \text{....on simplification}$$

$$= \frac{3(ay - x^2)}{3(y^2 - ax)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(ay - x^2)}{(y^2 - ax)} \quad \text{... (2)}$$

Now at point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ i.e. at $x = \frac{3a}{2}$ and $y = \frac{3a}{2}$

Equation (2) becomes,

$$\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{a\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)}$$

$$= \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}} \quad \text{... on simplification}$$

$$= \frac{a^2\left(\frac{3}{2} - \frac{9}{4}\right)}{a^2\left(\frac{9}{4} - \frac{3}{2}\right)} \quad \text{taking } a^2 \text{common}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{\frac{3\times 2}{4} - \frac{9}{4}}{\frac{9}{4} - \frac{3\times 2}{4}} = \frac{\frac{6-9}{4}}{\frac{9-6}{4}} = \frac{-3}{3} = -1$$

$$\therefore \left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -1 \checkmark \quad \text{...Ans.}$$

Ex. 10.1.40 S-2011, 4 Marks.

If $\sin y = x \sin(a + y)$ show that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

☑ **Soln. :**

Given : $\sin y = x \sin(a + y)$

$$\therefore \quad x = \frac{\sin y}{\sin(a + y)}$$

Differentiate w. r to y we get

$$\frac{dx}{dy} = \underbrace{\frac{d}{dy}\left[\frac{\sin y}{\sin(a + y)}\right]}_{\frac{d}{dy}\left(\frac{u}{v}\right) \text{ from}}$$

➔ **Using Quotient rule of derivative :**

$$\ldots\left[\frac{d}{dy}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dy}u - u\frac{d}{dy}v}{v^2} \text{ with } u = \sin y \text{ and } v = \sin(a + y)\right]$$

$$\therefore \Rightarrow \frac{dx}{dy} = \frac{\sin(a+y)\frac{d}{dy}\sin y - \sin y\frac{d}{dy}\sin(a+y)}{[\sin(a+y)^1]^2}$$

$$\left[\because \frac{d}{dy}\sin y = \cos y \text{ and } \frac{d}{dy}\sin(a+y) = \cos(a+y)\right]$$

$$\therefore \quad \frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)}$$

$\because$ $\sin(A - B) = \sin A \cos B - \cos A \sin B$

Here $A = a + y$ and $B = y$

$$\therefore \quad \frac{dx}{dy} = \frac{\sin[(a + y - y)]}{\sin^2(a + y)}$$

$$\therefore \quad \frac{dx}{dy} = \frac{\sin a}{\sin^2(a + y)} \quad \text{... on simplification}$$

$$\therefore \quad \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{\left(\frac{\sin a}{\sin^2(a+y)}\right)} \quad - \text{ Reciprocal of } \frac{dx}{dy}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a} \quad \left\{\because \frac{1}{a/x} = \frac{x}{a}\right. \checkmark \quad \text{...Ans.}$$

Ex. 10.1.41 S-2011, 4 Marks.

Diff $(\tan x)^{\cot x}$ w.r. to x

☑ **Soln. :**

Consider, $y = (\tan x)^{\cot x}$ ... (1)

Taking log on both sides, we get

$$\log y = \log(\tan x)^{\cot x}$$

➔ **Using property of logarithm** ...[$\log a^m = m \log a$]

$$\therefore \quad \log y = \cot x\,[\log(\tan x)]$$

Differentiate both sides w.r.to x, we get,

$$\frac{d}{dx}\log y = \underbrace{\frac{d}{dx}\{\cot x\,[\log \tan x]\}}_{\frac{d}{dx}(u \cdot v)\ form}$$

➔ **Using product rule of derivative :**

$$\ldots\left[\frac{d}{dx}uv = u\frac{d}{dx}v + v\frac{d}{dx}u\right]$$

Here u = cot x, v = log tan x

$$\therefore \underbrace{\frac{d}{dx}\log y}_{\frac{1}{y}\frac{dy}{dx}} = \cot x\underbrace{\frac{d}{dx}[\log \tan x]}_{\frac{1}{\tan x}\left(\frac{d}{dx}\tan x\right)} + \log \tan x\underbrace{\frac{d}{dx}(\cot x)}_{(-\operatorname{cosec}^2 x)}$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \cot x\frac{1}{\tan x}\underbrace{\left(\frac{d}{dx}\tan x\right)}_{\sec^2 x} + \log \tan x\,(-\operatorname{cosec}^2 x)$$

…By derivative result

$$\therefore \frac{1}{y}\frac{dy}{dx} = \cot x\underbrace{\frac{1}{\tan x}}_{\cot x}\underbrace{\sec^2 x}_{\frac{1}{\cos^2 x}} - \operatorname{cosec}^2 x\log \tan x \qquad \to \left\{\because \frac{1}{\cos x} = \sec x\right.$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \underbrace{\cot x \cdot \cot x}_{\cot^2 x}\frac{1}{\cos^2 x} - \operatorname{cosec}^2 x\log \tan x$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \underbrace{\cot^2 x}_{\frac{\cos^2 x}{\sin^2 x}}\frac{1}{\cos^2 x} - \operatorname{cosec}^2 x\log \tan x$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{\cos^2 x}{\sin^2 x}\,\frac{1}{\cos^2 x} - \operatorname{cosec}^2 x\log \tan x$$

$$\because \cot x = \frac{\cos x}{\sin x}$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \underbrace{\frac{1}{\sin^2 x}} - \operatorname{cosec}^2 x\log \tan x$$

$$= \mathit{\operatorname{cosec}^2 x - \operatorname{cosec}^2 x\log \tan x}$$

$$= \operatorname{cosec}^2 x\,[\,1 - \log \tan x\,]$$

… Taking $\operatorname{cosec}^2 x$ common

$$\therefore \frac{1}{y}\frac{dy}{dx} = \operatorname{cosec}^2 x\,[1 - \log \tan x]$$

$$\therefore \frac{dy}{dx} = y\,[\operatorname{cosec}^2 x\,(1 - \log \tan x)]$$

$$\therefore \mathbf{\frac{dy}{dx} = (\tan x)^{\cot x}[\,\operatorname{cosec}^2 x\,(1 - \log \tan x)\,]}\ ✓$$ **…Ans.**

Ex. 10.1.42 : Find if $y = x^y$, prove that $\frac{dy}{dx} = \frac{y^2}{x\,(1 - y\log x)}$

☑ **Soln. :**

Given : $y = x^y$ … (1)

Taking log on both sides, we get

$$\log y = \underbrace{\log x^y}_{y\log x}$$

➔ **Using property of logarithm** …[$\log a^m = m\log a$]

$$\therefore \log y = y\log x$$

Now differential both sides w. r to x we get

$$\frac{d}{dx}\log y = \underbrace{\frac{d}{dx}[\,y\log x\,]}_{\frac{d}{dx}(u\,.\,v)\ form}$$

➔ **Using product rule of derivative :**

$$\ldots\left[\frac{d}{dx}(u.\,v) = u\frac{d}{dx}v + v\frac{d}{dx}u\right]$$

$$\therefore \underbrace{\frac{d}{dx}\log y}_{\frac{1}{y}\frac{dy}{dx}} = y\underbrace{\frac{d}{dx}\log x}_{\frac{1}{x}} + \log x\frac{d}{dx}y$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = y\frac{1}{x} + \log x\frac{dy}{dx}$$ … by derivative rule

Now, $\frac{1}{y}\frac{dy}{dx} - \log x\frac{dy}{dx} = \frac{y}{x}$ collecting terms of $\frac{dy}{dx}$ on LHS

$$\left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$$

$$\therefore \left(\frac{1 - y\log x}{y}\right)\frac{dy}{dx} = \frac{y}{x}$$ …on simplification

$$\therefore \frac{dy}{dx} = \frac{y}{x}\,\frac{y}{1 - y\log x}$$

$$\therefore \mathbf{\frac{dy}{dx} = \frac{y^2}{x\,(1 - y\log x)}}\ ✓$$ **…Ans.**

Ex. 10.1.43 W-2011, S-2013, 4 Marks.

If $y = (\sin x)^{\log x}$, find $\frac{dy}{dx}$

☑ **Soln. :**

Given : $y = (\sin x)^{\log x}$ … (1)

Taking log on both sides we get

$$\log y = \underbrace{\log (\sin x)^{\log x}}_{\log x\,\log(\sin x)}$$

➔ **Using property of logarithm :** …[$\log a^m = m\log a$]

$$\therefore \log y = \log x\,\log(\sin x)$$

Now differentiate both sides w. r to x we get

$$\underbrace{\frac{d}{dx}\log y}_{\frac{1}{y}\frac{dy}{dx}} = \underbrace{\frac{d}{dx}[\,\log x\,\log \sin x\,]}_{\frac{d}{dx}(u.\,v)\ form}$$

➔ Using product rule of derivative :

$$\ldots\left[\frac{d}{dx}u\cdot v = u\frac{d}{dx}v + v\frac{d}{dx}u\right]$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \log x\underbrace{\frac{d}{dx}(\log \sin x)}_{\frac{1}{\sin x}\frac{d}{dx}\sin x} + \log \sin x\underbrace{\frac{d}{dx}\log x}_{1/x}$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \log x\frac{1}{\sin x}\underbrace{\frac{d}{dx}\sin x}_{\cos x} + \log \sin x\frac{1}{x}$$

...[by derivative rule]

$$\therefore \frac{1}{y}\frac{dy}{dx} = \log x\underbrace{\frac{1}{\sin x}\cos x}_{\cot x} + \log \sin x\cdot\frac{1}{x}$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \cot x\cdot\log x + \frac{1}{x}\cdot\log \sin x$$

$$\therefore \frac{dy}{dx} = y\left[\cot x\cdot\log x + \frac{1}{x}\log \sin x\right]$$

$$\therefore \frac{dy}{dx} = (\sin x)^{\log x}\left[\cot x\cdot\log x + \frac{1}{x}\log \sin x\right] \checkmark \quad \ldots\text{Ans.}$$

Ex. 10.1.44 S-2008, 4 Marks.

Differentiate $(\log x)^{\sin x}$, w.r. to x

☑ Soln. :

Consider $y = (\log x)^{\sin x}$ ... (1)

Taking log on both sides we get

$$\log y = \underbrace{\log\left[(\log x)^{\sin x}\right]}_{\sin x \log(\log x)}$$

➔ Using property of logarithm : ...[$\log a^m = m \log a$]

$$\therefore \log y = \sin x \log(\log x)$$

Differentiate both sides w. r. to x we get

$$\underbrace{\frac{d}{dx}\log y}_{\frac{1}{y}\frac{dy}{dx}} = \underbrace{\frac{d}{dx}[\sin x \log(\log x)]}_{\frac{d}{dx}(u.\,v)\text{ form}}$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \sin x\underbrace{\frac{d}{dx}[\log(\log x)]}_{\frac{1}{\log x}\frac{d}{dx}(\log x)} + \log(\log x)\underbrace{\frac{d}{dx}\sin x}_{\cos x}$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \sin x\cdot\frac{1}{\log x}\underbrace{\frac{d}{dx}(\log x)}_{\frac{1}{x}} + \log(\log x)\cdot\cos x$$

... [by derivative rule]

$$\therefore \frac{1}{y}\frac{dy}{dx} = \sin x\frac{1}{\log x\cdot x} + \log(\log x)\cos x$$

$$= \frac{\sin x}{x\log x} + \cos x\log(\log x)$$

$$\therefore \frac{dy}{dx} = y\left[\frac{\sin x}{x\log x} + \cos x\log(\log x)\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\sin x}\left[\frac{\sin x}{x\log x} + \cos x\log(\log x)\right] \checkmark \quad \ldots\text{Ans.}$$

Ex. 10.1.45 S-2015, 2 Marks.

If $x = a\sec t$ and $y = b\tan t$ then find $\frac{dy}{dx}$.

☑ Soln. :

Given : $x = a\sec t$

$y = b\tan t$

$x = a\sec t$	$y = b\tan t$
Differentiate both sides w.r.t. t,	Differentiate both sides w.r.t. t,
$\frac{dx}{dt} = \frac{d}{dt}(a\sec t)$	$\frac{dy}{dt} = \frac{d}{dt}(b\tan t)$
➔ Using property of derivative	**➔ Using property of derivative**
$\ldots\left[\frac{d}{dx}[k\,f(x)] = k\frac{d}{dx}f(x)\right]$	$\ldots\left[\frac{d}{dx}[k\,f(x)] = k\frac{d}{dx}f(x)\right]$
$\frac{dx}{dt} = a\frac{d}{dt}(\sec t)$	$\frac{dy}{dt} = b\frac{d}{dt}(\tan t)$
$\therefore \frac{dx}{dt} = a(\sec t\ \tan t)$ ...(1)	$\therefore \frac{dy}{dt} = b(\sec^2 t)$ ...(2)

We know, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ [by derivative of parametric functions]

Substitute values from Equations (1) and (2), we get,

$$\frac{dy}{dx} = \frac{b\sec^2 t}{a\sec t\tan t}$$

$$\therefore \frac{dy}{dx} = \frac{b\sec t}{a\tan t}$$

$$= \frac{b\frac{1}{\cos t}}{a\frac{\sin t}{\cos t}} = b\frac{1}{\cancel{\cos}\,t}\cdot\frac{\cancel{\cos}\,t}{a\sin t} \quad \left[\because \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b}\times\frac{d}{c}\right]$$

$$= \frac{b}{a}\cdot\frac{1}{\sin t}$$

$$\frac{dy}{dx} = \frac{b}{a\sin t} \quad \text{OR} \quad \frac{dy}{dx} = \left(\frac{b}{a}\right)\text{cosec } t \checkmark \quad \ldots\text{Ans.}$$

Ex. 10.1.46 S-2012, W-2013, W-2015, 4 Marks.

If $x = 3\cos\theta - 2\cos^3\theta$, $y = 3\sin\theta - 2\sin^3\theta$

find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$

☑ Soln. :

Step I :

Given : $x = 3\cos\theta - 2\cos^3\theta$

Differentiate both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}[3\cos\theta - 2\cos^3\theta]$$

➔ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}[a\,f(x) + b\,g(x)] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)\right]$$

$$= 3\frac{d}{d\theta}(\cos\theta) - 2\frac{d}{d\theta}(\cos^3\theta)$$

➔ **Using derivative of standard function**

$$\ldots\left[\frac{d}{dx}\cos x = -\sin x, \frac{d}{dx}\cos^n x = n\cos^{n-1}x\frac{d}{dx}\cos x\right]$$

$$\frac{dx}{d\theta} = 3(-\sin\theta) - 2\left[3\cos^2\theta\cdot\frac{d}{d\theta}(\cos\theta)\right]$$

$$\frac{dx}{d\theta} = -3\sin\theta - 2[3\cos^2\theta(-\sin\theta)]$$

$$\frac{dx}{d\theta} = -3\sin\theta + 6\cos^2\theta\sin\theta$$

….from RHS, common out $(-3\sin\theta)$ term we get,

$$\frac{dx}{d\theta} = -3\sin\theta[1 - 2\cos^2\theta] \quad \ldots(1)$$

Step II :

$$y = 3\sin\theta - 2\sin^3\theta \quad \text{(given)}$$

Differentiate both sides w.r.t. θ,

$$\frac{dy}{d\theta} = \frac{d}{d\theta}[3\sin\theta - 2\sin^3\theta]$$

➔ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}[a\,f(x) + b\,g(x)] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)\right]$$

$$= 3\frac{d}{d\theta}(\sin\theta) - 2\frac{d}{d\theta}(\sin^3\theta)$$

➔ **Using derivative of standard function**

$$\ldots\left[\frac{d}{dx}\sin x = \cos x;\ \frac{d}{dx}\sin^n x = n\sin^{n-1}x\cdot\frac{d}{dx}\sin x\right]$$

$$\frac{dy}{d\theta} = 3\cos\theta - 2\left[3\sin^2\theta\frac{d}{d\theta}(\sin\theta)\right]$$

$$\frac{dy}{d\theta} = 3\cos\theta - 2[3\sin^2\theta(\cos\theta)]$$

$$= 3\cos\theta - 6\sin^2\theta\cos\theta$$

Common out from RHS $(3\cos\theta)$ terms, we get

$$\frac{dy}{d\theta} = 3\cos\theta(1 - 2\sin^2\theta) \quad \ldots(2)$$

Step III : We know, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

[by derivative of parametric functions]

Substitute values from Equations (1) and (2), we get

$$\frac{dy}{dx} = \frac{\cancel{3}\cos\theta(1 - 2\sin^2\theta)}{-\cancel{3}\sin\theta(1 - 2\cos^2\theta)}$$

$$= \frac{\cos\theta(1 - 2\sin^2\theta)}{-\sin\theta[-(2\cos^2\theta - 1)]}$$

➔ **Using standard trigonometric formulae**

$$\ldots\left[\begin{matrix}\cos 2\theta = 2\cos^2\theta - 1\\ \text{and } \cos 2\theta = 1 - 2\sin^2\theta\end{matrix}\right]$$

$$= \frac{\cos\theta(\cancel{\cos 2\theta})}{-\sin\theta(-\cancel{\cos 2\theta})}$$

$$= \frac{\cos\theta}{+\sin\theta}$$

$$\therefore \frac{dy}{dx} = \cot\theta \quad \ldots(3)$$

Step IV :

At $\theta = \frac{\pi}{4}$

i.e. Put $\theta = \frac{\pi}{4}$ in Equation (3)

$$\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = \cos\left(\frac{\pi}{4}\right) = 1$$

➔ **Using standard trigonometric value** …$[\cos\frac{\pi}{4} = 1]$

$$\therefore \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = 1 \ ✓ \quad \text{...Ans.}$$

Ex. 10.1.47 W-2015, 2 Marks.

Find $\frac{dy}{dx}$ if $x = 4\sin 3\theta$, $y = 4\cos 6\theta$.

☑ **Soln. :**

Given : $x = 4\sin 3\theta$

Differentiate w.r.t θ,

$$\therefore \frac{dx}{d\theta} = \frac{d}{d\theta}[4\sin 3\theta]$$

➔ **Using derivative of function :**

$$\ldots\left[\frac{d}{dx}\sin ax = \cos ax\cdot\frac{d}{dx}(ax)\right]$$

$$= 4\cos 3\theta\cdot\underbrace{\frac{d}{d\theta}(3\theta)}_{3} = 4\cos 3\theta\,(3)$$

$$\therefore \frac{dx}{d\theta} = 12\cos 3\theta \quad \ldots(1)$$

Now, $y = 4\cos 6\theta$

Differentiate w.r.t. θ,

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(4\cos 6\theta)$$

$$= 4\underbrace{\frac{d}{d\theta}(\cos 6\theta)}_{-\sin 6\theta} = 4[-\sin 6\theta]\underbrace{\frac{d}{d\theta}(6\theta)}_{6}$$

$$= -4\sin 6\theta\cdot(6)$$

$$\frac{dy}{d\theta} = -24\sin 6\theta \quad \ldots(2)$$

We know,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \quad \text{…(by derivative of parametric functions)}$$

…[Substitute values from Equations (1) and (2)] we get,

$$= \frac{-24 \sin 6\theta}{12 \cos 3\theta}$$

$$\frac{dy}{dx} = \frac{-2 \sin 6\theta}{\cos 3\theta} \checkmark \quad \text{...Ans.}$$

Ex. 10.1.48 W-2016, 2 Marks.

Find $\frac{dy}{dx}$, If $x = 3 \sin 4\theta$, $y = 4 \cos 3\theta$

Soln. : $x = 3 \sin 4\theta$, $y = 4 \cos 3\theta$

Differentiate both sides w.r.t. x we get,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(3 \sin 4\theta) \qquad \frac{dy}{d\theta} = \frac{d}{d\theta}(4 \cos 3\theta)$$

$$\frac{dx}{d\theta} = 3\frac{d}{d\theta}(\sin 4\theta) \qquad \frac{dy}{d\theta} = 4\frac{d}{d\theta}\cos 3\theta$$

$$= 3 (\cos 4\theta)(4) \qquad \frac{dy}{d\theta} = 4(-\sin 3\theta)(3)$$

➔ **Using linearity property of derivative :**

$$\ldots\left[\frac{d}{dx}[a f(x) + b g(x)] = a\frac{d}{dx}(f(x)) + b\frac{d}{dx}(g(x))\right]$$

$$\frac{dx}{d\theta} = 12 \cos 4\theta \quad \text{...(1)}$$

$$\frac{dy}{d\theta} = -12 \sin 3\theta \quad \text{...(2)}$$

$$\because \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Substituting values from Equations (1) and (2),

$$= \frac{-\cancel{12} \sin 3\theta}{\cancel{12} \cos 4\theta}$$

$$\frac{dy}{dx} = \frac{-\sin 3\theta}{\cos 4\theta} \checkmark \quad \text{...Ans.}$$

Ex. 10.1.49 W-2007, W-2011, W-2012, 4 Marks.

Find $\frac{dy}{dx}$, if $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$.

☑ **Soln. :**

Step I :

Given : $x = a(\cos t + t \sin t)$

Differentiate both sides, w.r. to t,

$$\frac{dx}{dt} = \frac{d}{dt}[a(\cos t + t \sin t)]$$

➔ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}[a f(x) + b g(x)] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)\right]$$

$$= a\left[\frac{d}{dt}(\cos t) + \frac{d}{dt}(t \sin t)\right]$$

➔ **Using product rule of derivative**

[for II[nd] term of RHS]

$$\ldots\left[\frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$= a\left[\frac{d}{dt}(\cos t) + \left[t \cdot \frac{d}{dt}(\sin t) + \sin t \cdot \frac{d}{dt}(t)\right]\right]$$

➔ **Using derivative of standard function**

$$\ldots\left[\frac{d}{dt}(\cos t) = -\sin t ; \frac{d}{dt}(t) = 1\right]$$

$$\frac{dx}{dt} = a[(-\sin t) + (t \cos t + (1) \sin t)]$$

$$= a[-\cancel{\sin} t + t \cos t + \cancel{\sin} t]$$

...(by simplification)

$$\frac{dx}{dt} = a[t \cos t]$$

$$\therefore \frac{dx}{dt} = a t \cos t \quad \text{...(1)}$$

Step II :

$$y = a(\sin t - t \cos t) \quad \text{(given)}$$

Differentiate both sides w.r.t. t,

$$\therefore \frac{dy}{dt} = \frac{d}{dt}[a(\sin t - t \cos t)]$$

➔ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}[a f(x) + b g(x)] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)\right]$$

$$= a\frac{d}{dt}(\sin t) - \frac{d}{dt}(t \cos t)$$

➔ **Using product rule of derivative**

$$\left[\text{for II}^{\text{nd}} \text{ term: } \frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$\frac{dy}{dt} = a\{(\cos t) - [t(-\sin t) + (1)\cos t]\}$$

$$= a[\cancel{\cos} t + t \sin t - \cancel{\cos} t]$$

(by simplification)

$$\therefore \frac{dy}{dt} = a(t \sin t)$$

$$\frac{dy}{dt} = at \sin t \quad \text{...(2)}$$

Step III :

$$\text{We know, } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

[by derivative of parametric functions]

Substitute values from Equations (1) and (2), we get,

$$\frac{dy}{dx} = \frac{at \cancel{\sin} t}{at \cancel{\cos} t}$$

$$\therefore \frac{dy}{dx} = \tan t \checkmark \quad \text{...Ans.}$$

Ex. 10.1.50 **W-2008, S-2010, S-2011, W-2015, 4 Marks.**

If x = a (θ + sin θ), y = a (1 – cos θ) find $\frac{dy}{dx}$.

☑ **Soln. :**

Step I :

Given : $x = a(\theta + \sin\theta)$

Differentiate both sides w.r. to θ,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}[a(\theta + \sin\theta)].$$

➔ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}[a\,f(x) + b\,g(x)] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)\right]$$

$$= a\left[\frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(\sin\theta)\right]$$

$$\frac{dx}{d\theta} = a(1 + \cos\theta) \qquad \ldots(1)$$

Step II :

$y = a(1 - \cos\theta)$

Differentiate both sides w.r. to θ,

$$\frac{dy}{d\theta} = \frac{d}{d\theta}[a(1 - \cos\theta)]$$

➔ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}[a\,f(x) + b\,g(x)] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)\right]$$

$$= a\left[\frac{d}{d\theta}(1) - \frac{d}{d\theta}(\cos\theta)\right]$$

$$\frac{dy}{d\theta} = a[0 - (-\sin\theta)]$$

$$\frac{dy}{d\theta} = a\sin\theta \qquad \ldots(2)$$

Step III :

We know, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ [Derivative of parametric functions]

Substitute values from Equations (1) and (2), we get,

$$\frac{dy}{dx} = \frac{\not{a}\sin\theta}{\not{a}(1 + \cos\theta)}$$

$$\therefore \frac{dy}{dx} = \frac{\sin\theta}{1 + \cos\theta}$$

➔ **Using standard trigonometric formulae**

$$\ldots\left[\text{half angle formulae } \sin\theta = 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2} \text{ and } \frac{1 + \cos\theta}{2} = \cos^2\left(\frac{\theta}{2}\right)\right]$$

Note that, $\frac{dy}{dx} = \frac{\not{2}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\not{2}\cos^2\left(\frac{\theta}{2}\right)} = \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$

$$\therefore \frac{dy}{dx} = \tan\left(\frac{\theta}{2}\right) \checkmark \qquad \ldots\text{Ans.}$$

Ex. 10.1.51 **S-2008, W-2009, S-2013, W-2014, 4 Marks.**

Find $\frac{dy}{dx}$, if x = a (θ – sin θ), y = a (1 – cos θ).

Also find $\frac{d^2y}{dx^2}$ At $\theta = \frac{\pi}{4}$

☑ **Soln. :**

Step I :

Given : $x = a(\theta - \sin\theta)$

Differentiate both sides w.r.t. θ,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}[a(\theta - \sin\theta)]$$

➔ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}[a\,f(x) + b\,g(x)] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)\right]$$

$$= a\left[\underbrace{\frac{d}{d\theta}(\theta)}_{1} - \underbrace{\frac{d}{d\theta}(\sin\theta)}_{\cos\theta}\right]$$

$$\frac{dx}{d\theta} = a[1 - \cos\theta] \qquad \ldots(1)$$

Step II :

Also, $y = a(1 - \cos\theta)$ (given)

Differentiate both sides w.r.t. θ,

$$\frac{dy}{d\theta} = \frac{d}{d\theta}[a(1 - \cos\theta)]$$

➔ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}[a\,f(x) + b\,g(x)] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)\right]$$

$$= a\left[\underbrace{\frac{d}{d\theta}(1)}_{1} - \underbrace{\frac{d}{d\theta}\cos\theta}_{\cos\theta}\right]$$

$$\frac{dy}{d\theta} = a(0 - (-\sin\theta))$$

$$\frac{dy}{d\theta} = a\sin\theta \qquad \ldots(2)$$

Step III :

We know, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ [Derivative of parametric functions]

Substitute values from Equations (1) and (2), we get,

$$\therefore \frac{dy}{dx} = \frac{\not{a}\sin\theta}{\not{a}(1-\cos\theta)}$$

$$\therefore \frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta} \qquad \text{…(3)}$$

➔ Using standard trigonometric formulae

$$\text{…}\left[\text{Half angle formulae } \sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \text{ and } \frac{1-\cos\theta}{2} = \sin^2\left(\frac{\theta}{2}\right)\right]$$

Note this, $$\frac{dy}{dx} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)} = \frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

$$\therefore \frac{dy}{dx} = \cot\left(\frac{\theta}{2}\right)$$

Step IV :

From Equation (3),

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$$

Differentiate again w.r.to x, it gives,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\sin\theta}{1-\cos\theta}\right)$$

➔ Using Quotient rule of derivative

$$\text{…}\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\cdot\frac{du}{dx} - u\cdot\frac{dv}{dx}}{v^2}\right]$$

$$\left[\text{with } u = \sin\theta \text{ and } v = 1-\cos\theta\right]$$

$$= \frac{(1-\cos\theta)\frac{d}{dx}(\sin\theta) - \sin\theta\frac{d}{dx}(1-\cos\theta)}{(1-\cos\theta)^2}$$

(by quotient rule)

➔ Using derivative and composite function

$$\text{…}\left[\frac{d}{dx}f(\theta) = \frac{d}{d\theta}f(\theta)\cdot\frac{d\theta}{dx}\right]$$

$$= \frac{(1-\cos\theta)(\cos\theta)\frac{d\theta}{dx} - \sin\theta\,(0-(-\sin\theta))\cdot\frac{d\theta}{dx}}{(1-\cos\theta)^2}$$

$$\left[\text{simplify and common out } \frac{d\theta}{dx} \text{ from numerator of R.H.S.}\right]$$

$$= \left[\frac{\cos\theta - \cos^2\theta - \sin^2\theta}{(1-\cos\theta)^2}\right]\frac{d\theta}{dx} = \frac{\cos\theta - [\cos^2\theta + \sin^2\theta]}{(1-\cos\theta)^2}\cdot\frac{1}{\left(\frac{dx}{d\theta}\right)}$$

Substitute value from Equation (1),

➔ Using standard trigonometric formulae

…[$\sin^2\theta + \cos^2\theta = 1$]

$$= \frac{\cos\theta - 1}{(1-\cos\theta)^2}\cdot\frac{1}{a(1-\cos\theta)}$$

$$= \frac{-[1-\cos\theta]}{(1-\cos\theta)^2}\cdot\frac{1}{a(1-\cos\theta)}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{a(1-\cos\theta)^2} \qquad \text{…(4)}$$

Step V : At $\theta = \frac{\pi}{4}$, i.e. put $\theta = \frac{\pi}{4}$ in Equation (3),

$$\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = \frac{\sin\left(\frac{\pi}{4}\right)}{1-\cos\left(\frac{\pi}{4}\right)}$$

➔ Using standard trigonometric value

$$\text{…}\left[\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \text{ and } \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\right]$$

$$= \frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{2}-1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}-1} = \frac{1}{\sqrt{2}-1}$$

$$\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = \frac{1}{\sqrt{2}-1}$$

From Equation (4),

$$\left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{4}} = \frac{1}{a\left(1-\cos\frac{\pi}{4}\right)^2} = \frac{-1}{a\left(1-\frac{1}{\sqrt{2}}\right)^2} = \frac{-1}{\frac{a(\sqrt{2}-1)^2}{(\sqrt{2})^2}}$$

$$= \frac{(-1)(\sqrt{2})^2}{a(\sqrt{2}-1)^2} = \frac{-2}{a[2-2\sqrt{2}+1]} = \frac{-2}{a[3-2\sqrt{2}]}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{4}} = \frac{-2}{a[3-2\sqrt{2}]} \checkmark \qquad \text{…Ans.}$$

Ex. 10.1.52 S-2010, 4 Marks.

Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w.r.t. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

☑ **Soln. :**

Step I :

Consider, $$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

and $$z = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

By observing both y and z there is a suitable substitution as x = tan θ.

$$\left.\begin{aligned} \text{Put } x &= \tan\theta \\ \therefore \theta &= \tan^{-1}x \end{aligned}\right\} \qquad \text{…(1)}$$

$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$y = \sin^{-1}(\sin 2\theta)$

$y = 2\theta$

$\therefore y = 2\tan^{-1}x$ [From (1)]

Differentiate w.r.t. x,

$\frac{dy}{dx} = \frac{d}{dx}[2\tan^{-1}x]$

$\frac{dy}{dx} = 2\frac{1}{1+x^2}$

$z = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$z = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$z = \cos^{-1}(\cos 2\theta)$

$z = 2\theta$

$z = 2\tan^{-1}x$ [From (1)]

Differentiate w.r.t. x,

$\frac{dz}{dx} = \frac{d}{dx}[2\tan^{-1}x]$

$\frac{dz}{dx} = 2\frac{1}{1+x^2}$

Step II :

We know, $\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$ [To find derivative w.r.t. z]

$\frac{dy}{dz} = \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}}$ (From above values)

$\therefore \frac{dy}{dz} = 1$ ✓ ...Ans.

Ex. 10.1.53 : Differentiate $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ w.r.t. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

☑ **Soln. :**

Step I : Consider, $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ and $z = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

By observing both y and z there is a suitable substitution as x = tan θ.

Put $x = \tan\theta$
$\therefore \theta = \tan^{-1}x$ ... (1)

$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\therefore y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$y = \cos^{-1}(\cos 2\theta)$

$y = 2\theta$

$y = 2\tan^{-1}x$

Differentiate w.r.t. x,

$\frac{dy}{dx} = \frac{d}{dx}[2\tan^{-1}x]$

$\frac{dy}{dx} = 2\frac{1}{1+x^2}$

$z = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$z = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$z = \sin^{-1}(\sin 2\theta)$

$z = 2\theta$

$z = 2\tan^{-1}x$

Differentiate w.r.t. x,

$\frac{dz}{dx} = \frac{d}{dx}[2\tan^{-1}x]$

$\frac{dz}{dx} = 2\cdot\frac{1}{1+x^2}$

Step II :

We know, $\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$ [To find differentiate y w.r.t. z]

$\frac{dy}{dz} = \frac{2\cdot\frac{1}{1+x^2}}{2\cdot\frac{1}{1+x^2}}$ (From above values)

$\therefore \frac{dy}{dz} = 1$ ✓ ...Ans.

Ex. 10.1.54 .W-2010, 4 Marks.

Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

☑ **Soln. :**

Step I :

Consider, $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

and $z = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

By observing both y and z there is a suitable substitution as x = tan θ.

Put $x = \tan\theta$
$\therefore \theta = \tan^{-1}x$ ...(1)

$y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$

$y = \tan^{-1}\left(\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}\right)$

$y = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$

$y = \tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right)$

$y = \tan^{-1}\left(\frac{\frac{1-\cos\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\right)$

$y = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$z = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$z = \sin^{-1}(\sin 2\theta)$ (by formula)

$z = 2\theta$

$z = 2\tan^{-1}x$ (From (1))

Differentiate w.r.t. x,

$\frac{dz}{dx} = \frac{d}{dx}(2\tan^{-1}x)$

$\frac{dz}{dx} = 2\frac{1}{1+x^2}$

Using half angle formulae

$y = \tan^{-1}\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$

$y = \tan^{-1}\left(\tan\left(\frac{\theta}{2}\right)\right) = \frac{\theta}{2} = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}x$

Differentiate w.r.t. x,

$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{2}\tan^{-1}x\right] = \frac{1}{2}\cdot\frac{1}{1+x^2}$

Step II : We know, $\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$ [To find derivative of y w.r.to z]

$$\frac{dy}{dz} = \frac{\frac{1}{2}\cdot\frac{1}{1+x^2}}{2\cdot\frac{1}{1+x^2}} = \frac{\frac{1}{2}}{2} \quad \text{(From above values)}$$

$$\therefore \frac{dy}{dz} = \frac{1}{4} \checkmark$$ …Ans.

Ex. 10.1.55 **S-2016, 4 Marks.**

Differentiate $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$ with respect to $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$ (4 Marks)

☑ **Soln. :**

Step I : Consider, $y = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$

and $z = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$ …(1)

By observing both y and z there is a suitable substitution as x = sin θ.

Put $x = \sin\theta \Rightarrow \therefore \theta = \sin^{-1}x$ …(2)

Using substitution in Equation (1), we get

$\therefore y = \cos^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$y = \cos^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$

$y = \cos^{-1}(2\sin\theta \cdot \cos\theta)$

$y = \cos^{-1}(\sin 2\theta)$

$= \cos^{-1}\left[\cos\left(\frac{\pi}{2} - 2\theta\right)\right]$

$y = \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\sin^{-1}x$

…[From Equation (2)]

Differentiate w.r.to x,

$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\pi}{2} - 2\sin^{-1}x\right]$

$= 0 - 2\frac{1}{\sqrt{1-x^2}}$

$\frac{dy}{dx} = 2\frac{1}{\sqrt{1-x^2}}$ …(3)

$\therefore z = \sec^{-1}\left(\frac{1}{\sqrt{1-\sin^2\theta}}\right)$

$z = \sec^{-1}\left(\frac{1}{\sqrt{\cos^2\theta}}\right)$

$z = \sec^{-1}\left(\frac{1}{\cos\theta}\right)$

$z = \sec^{-1}(\sec\theta)$

$z = \theta$

$z = \sin^{-1}x$

…[From Equation (2)]

Differentiate w.r.to x,

$\frac{dz}{dx} = \frac{1}{\sqrt{1-x^2}}$ …(4)

Step II :

We know, $\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$ [To find derivative of y w. r. to z]

$$\frac{dy}{dz} = \frac{-2\frac{1}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}}$$ …Values from Equation (3) and (4),

$$\therefore \frac{dy}{dz} = -2 \checkmark$$ …Ans.

Ex. 10.1.56 **S-2014, 4 Marks.**

Differentiate $\cos^{-1}(2x^2-1)$ w.r.t. $\sin^{-1}(2x\sqrt{1-x^2})$

☑ **Soln. :**

Step I : Consider, $y = \cos^{-1}(2x^2-1)$ and $z = \sin^{-1}(2x\sqrt{1-x^2})$

By observing both y and z there is a suitable substitution as x = cos θ

Put $x = \cos\theta$, $\theta = \cos^{-1}x$ …(2)

$\therefore y = \cos^{-1}(2\cos^2\theta - 1)$

$y = \cos^{-1}(\cos 2\theta)$

(by formula of cos 2θ)

$y = 2\theta$

$y = 2\cos^{-1}x$

[from (1)] $z = 2\theta$

Differentiate w.r. to x,

$\frac{dy}{dx} = \frac{d}{dx}[2\cos^{-1}x]$

$\frac{dz}{dx} = \frac{d}{dx}[2\cos^{-1}x]$

$\frac{dy}{dx} = 2\frac{-1}{\sqrt{1-x^2}}$

$z = \sin^{-1}(2\cos\theta\sqrt{1-\cos^2\theta})$

$= \sin^{-1}(2\cos\theta\cdot\sqrt{\sin^2\theta})$

$= \sin^{-1}(2\cos\theta\sin\theta)$

$z = \sin^{-1}(\sin 2\theta)$

(by formula of sin 2θ)

$z = 2\cos^{-1}x$

Differentiate w.r. to x,

$\frac{dz}{dx} = 2\frac{-1}{\sqrt{1-x^2}}$

Step II : We know,

$$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} \text{ [To find derivative of y w. r.t. z]}$$

$$\frac{dy}{dz} = \frac{2\frac{-1}{\sqrt{1-x^2}}}{2\frac{-1}{\sqrt{1-x^2}}} \quad \text{(value from above)}$$

$$\therefore \frac{dy}{dz} = 1 \checkmark$$ …Ans.

Chapter Ends…

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Chapter 11 : APPLICATIONS OF DERIVATIVE

EXERCISE 11.1

Ex. 11.1.1: Find the equation of the tangent and normal to the curve $X^2 + 3xy + y^2 = 5$ at (1, 1)

Soln. :

Step I : We know the equation of the tangent passing through point (x_1, y_1) is

$$y - y_1 = m(x - x_1) \quad ...(1)$$

where m is slope of the tangent

$$\text{and } m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

first find $\frac{dy}{dx}$ at point (1, 1)

Step II : Given equation of the curve is

$$x^2 + 3xy + y^2 = 5$$

Differentiate both sides w.r to x we get

$$\frac{d}{dx}[x^2 + 3xy + y^2] = \frac{d}{dx}(5)$$

➢ **Using linearity property of derivative**

$$...\left[\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\right]$$

$$\underbrace{\frac{d}{dx}x^2}_{2x} + \underbrace{\frac{d}{dx}(3xy)}_{3\frac{d}{dx}(xy)} + \underbrace{\frac{d}{dx}y^2}_{2y\frac{dy}{dx}} = \underbrace{\frac{d}{dx}5}_{0}$$

$$\therefore\ 2x + 3\underbrace{\frac{d}{dx}(xy)}_{\frac{d}{dx}(uv)\text{ form}} + 2y\frac{dy}{dx} = 0$$

➢ **Using product rule of derivative**

$$...\left[\frac{d}{dx}(uv) = u\frac{d}{dx}v + v\frac{d}{dx}u\right]$$

$$\therefore\ 2x + 3\left[x\frac{dy}{dx} + y\underbrace{\frac{d}{dx}x}_{1}\right] + 2y\frac{dy}{dx} = 0$$

$$\therefore\ 2x + 3\left[x\frac{dy}{dx} + y(1)\right] + 2y\frac{dy}{dx} = 0$$

$$\therefore\ 2x + 3x\frac{dy}{dx} + 3y + 2y\frac{dy}{dx} = 0$$

...On simplification

$$\therefore\ 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - 3y$$

...Collecting common terms

$$\therefore\ (3x + 2y)\frac{dy}{dx} = \underbrace{-2x - 3y}_{-(2x+3y)}$$

$$\therefore\ (3x + 2y)\frac{dy}{dx} = -(2x + 3y)$$

$$\therefore \frac{dy}{dx} = \frac{-(2x + 3y)}{(3x + 2y)} \quad \text{...on simplification ...(2)}$$

Now slope of tangent at (1, 1) = $m = \left(\frac{dy}{dx}\right)_{(1,1)}$

$\therefore$ Equation (2) becomes $\left(\frac{dy}{dx}\right)_{(1,1)} = \frac{-[2(1) + 3(1)]}{[3(1) + 2(1)]}$

$$\begin{bmatrix}\text{Put } x = 1,\ y = 1 \\ \text{in Equation (2)}\end{bmatrix}$$

$$= -\left(\frac{2+3}{3+2}\right) = -\left(\frac{5}{5}\right) = -1$$

$$\therefore\ \underbrace{\left(\frac{dy}{dx}\right)_{(1,1)}}_{m} = -1$$

$$\therefore \text{ Slope of tangent } m = -1$$

Step III : Substitute this value of $m = -1$, $x_1 = 1$ and $y_1 = 1$ in Equation (1) we get

$\therefore$ Equation (1) becomes,

$$(y - 1) = \underbrace{(-1)(x - 1)}_{-x+1}$$

$$\therefore\ y - 1 = -x + 1$$

$$\therefore\ x + y = 1 + 1 \quad \text{...on simplification}$$

$$\therefore\ \mathbf{x + y = 2}\ ✓ \quad \text{...Ans.}$$

This is required equation of tangent.

Ex. 11.1.2 **(W-2013, 4 Marks)**

Find the equation of the tangent and normal to the curve $13x^3 + 2x^2y + y^3 = 1$ at (1, – 2).

Soln. :

Step I : We know, equation of tangent passing through (x_1, y_1) is

$$y - y_1 = m(x - x_1) \quad ...(1)$$

where m is slope of tangent and $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

Step II : Given equation of curve is

$$13x^3 + 2x^2y + y^3 = 1$$

Differentiate w.r.t. x, it gives

$$\frac{d}{dx}[13x^3 + 2x^2y + y^3] = \underbrace{\frac{d}{dx}(1)}_{0}$$

➢ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\right]$$

$$\frac{d}{dx}(13x^3) + \frac{d}{dx}(2x^2y) + \frac{d}{dx}y^3 = 0$$

➢ **Using property of derivative**

$$\ldots\left[\frac{d}{dx}k\,f(x) = k\frac{d}{dx}f(x)\right]$$

$$13\underbrace{\frac{d}{dx}x^3}_{3x^{3-1}} + 2\underbrace{\frac{d}{dx}(x^2y)}_{\frac{d}{dx}(u\cdot v)} + \underbrace{\frac{d}{dx}y^3}_{3y^{3-1}\frac{d}{dx}y} = 0$$

➢ **Using product rule of derivative**

$$\ldots\left[\begin{array}{l}\frac{d}{dx}x^n = n\,x^{n-1} \\ \text{and } \frac{d}{dx}u\cdot v = u\frac{d}{dx}v + v\frac{d}{dx}u\end{array}\right]$$

$$\therefore\ 13(3x^{3-1}) + 2\left(x^2\underbrace{\frac{d}{dx}y}_{\frac{dy}{dx}} + y\underbrace{\frac{d}{dx}x^2}_{2x}\right) + 3y^{3-1}\frac{d}{dx}y = 0$$

$$\therefore\ 13[3x^2] + 2\left[x^2\frac{dy}{dx} + y(2x)\right] + 3y^2\frac{dy}{dx} = 0$$

$$\therefore\ 39x^2 + 2x^2\frac{dy}{dx} + 4xy + 3y^2\frac{dy}{dx} = 0$$

...Simplifying terms

Now collecting terms of $\frac{dy}{dx}$ on LHS and remaining on RHS, we get

$$\underbrace{2x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx}}_{(2x^2+3y^2)\frac{dy}{dx}} = \underbrace{-39x^2 - 4xy}_{-(39x^2+4xy)}$$

$$\therefore\ \frac{dy}{dx} = \frac{-(39x^2+4xy)}{(2x^2+3y^2)} \quad \ldots(2)$$

Now at point (1, – 2) i.e. at x = 1 and y = – 2 we get,

$$\text{Equation (2)} \Rightarrow \left(\frac{dy}{dx}\right)_{(1,-2)} = \frac{-[39(1)^2 + 4(1)(-2)]}{2(1)^2 + 3(-2)^2}$$

...Replace x by 1 and y by – 2

$$\therefore\ \left(\frac{dy}{dx}\right)_{(1,-2)} = \frac{-[39-8]}{2+3(4)} \quad \text{...On simplification}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{(1,-2)} = \frac{-(31)}{2+12} = \frac{-31}{14}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{(1,-2)} = \frac{-31}{14}$$

$\therefore$ Slope of tangent at point (1, – 2) is

$$m = \left(\frac{dy}{dx}\right)_{(1,-2)} = \frac{-31}{14}$$

$$\left\{\because \frac{dy}{dx} = \text{slope of tangent} = m\right.$$

Step III : Substitute this value of $m = \frac{-31}{14}$; $x_1 = 1$, $y_1 = -2$ in Equation (1) we get

$$\text{Equation (1)} \Rightarrow \underbrace{y-(-2)}_{y+2} = \frac{-31}{14}(x-1)$$

$$\therefore\ y + 2 = \frac{-31}{14}(x-1)$$

$$\therefore\ 14(y+2) = -31(x-1)$$

$$\therefore\ 14y + 28 = -31x + 31 \quad \text{...On simplification}$$

$$\therefore\ 31x + 14y = \underbrace{31 - 28}_{3} \quad \text{...Collecting common terms}$$

$$\therefore\ \mathbf{31x + 14y = 3}$$

This is required equation of tangent.

Step IV : To find equation of normal,

We know

$$\text{Slope of normal} = \frac{-1}{\text{Slope of tangent}}$$

$$\text{i.e. } m_1 = \frac{-1}{m}$$

$$\therefore\ m_1 = \frac{-1}{(-31/14)} = -\left(\frac{-14}{31}\right) = \frac{14}{31}$$

$$\therefore\ m_1 = \frac{14}{31} \quad \text{i.e. slope of normal} = \frac{14}{31}$$

$\therefore$ Equation of normal passing through the point (1, – 2) with slope $m_1 = \frac{14}{31}$ is

$$\underbrace{y - y_1}_{y-(-2)} = \underbrace{m_1(x-x_1)}_{\frac{14}{31}(x-1)} \qquad \because m_1 = \frac{14}{31},\ x_1 = 1 \text{ and } y_1 = -2$$

$$\therefore\ y + 2 = \frac{14}{31}(x-1)$$

$$\therefore\ 31(y+2) = 14(x-1)$$

$$\therefore\ 31y + 62 = 14x - 14 \quad \text{...by simplification}$$

$$\therefore\ -14x + 31y = \underbrace{-14 - 62}_{-76} \quad \text{...Collecting common terms}$$

$$\therefore\ -14x + 31y = -76$$

$-(14x-31y) = -76$ ...Taking –ve sign common

$\therefore$ **(14x – 31y) = 76✓** **...Ans.**

This is the required equation of normal.

Ex. 11.1.3 (W-2014, S-2016. 4 Marks)

Show that equation of the tangent to the curve $\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 2$ at the point (a, b) is $\frac{x}{a}+\frac{y}{b} = 2$.

☑ **Soln. :**

Step I : We know, equation of tangent passing through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1) \quad ...(1)$$

where m is slope of the tangent and $m\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

➢ **Using law of indices**

$$\ldots\left[\left(\frac{a}{b}\right)^m = \left(\frac{a^m}{b^m}\right)\right]$$

Step II : Given, equation of the curve is

$$\underbrace{\left(\frac{x}{a}\right)^m}_{\frac{x^m}{a^m}} + \underbrace{\left(\frac{y}{b}\right)^m}_{\frac{y^m}{b^m}} = 2$$

$$\therefore \frac{x^m}{a^m} + \frac{y^m}{b^m} = 2$$

Differentiate both sides w.r.t. x we get

$$\underbrace{\frac{d}{dx}\left(\frac{x^m}{a^m}+\frac{y^m}{b^m}\right)}_{\frac{d}{dx}\left(\frac{x^m}{a^m}\right)+\frac{d}{dx}\left(\frac{y^m}{b^m}\right)} = \underbrace{\frac{d}{dx}2}_{0}$$

➢ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \text{ and } \frac{d}{dx}k = 0\right]$$

$$\therefore \underbrace{\frac{d}{dx}\left(\frac{x^m}{a^m}\right)}_{\frac{1}{a^m}\frac{d}{dx}x^m} + \underbrace{\frac{d}{dx}\left(\frac{y^m}{b^m}\right)}_{\frac{1}{b^m}\frac{d}{dx}y^m} = 0$$

➢ **Using property of derivative**

$$\ldots\left[\frac{d}{dx}k\,f(x) = k\frac{d}{dx}f(x)\right]$$

$$\therefore \frac{1}{a^m}\underbrace{\frac{d}{dx}x^m}_{m\,x^{m-1}} + \frac{1}{b^m}\underbrace{\frac{d}{dx}y^m}_{my^{m-1}\frac{d}{dx}y} = 0$$

$$\therefore \frac{1}{a^m}(mx^{m-1}) + \frac{1}{b^m}\left(my^{m-1}\frac{dy}{dx}\right) = 0$$

...by rule of derivative

$$\therefore \frac{my^{m-1}}{b^m} + \frac{my^{m-1}}{b^m}\frac{dy}{dx} = 0$$

...On simplification of term

$$\therefore \frac{my^{m-1}}{b^m}\frac{dy}{dx} = -\frac{mx^{m-1}}{a^m} \quad \text{...shifting term towards RHS}$$

$$\therefore \frac{dy}{dx} = -\frac{\cancel{m}x^{m-1}}{a^m}\times\frac{b^m}{\cancel{m}y^{m-1}} \quad \ldots\left\{\because \frac{a}{b}x = y \Rightarrow x = \frac{b}{a}y\right.$$

$$\therefore \frac{dy}{dx} = -\frac{b^m x^{m-1}}{a^m y^{m-1}}$$

Now at point (a, b) i.e. at x = a and y = b

$$\left(\frac{dy}{dx}\right)_{(a,b)} = \underbrace{\frac{-b^m(a)^{m-1}}{a^m(b)^{m-1}}}$$

$$= \frac{-\cancel{b^m}\,\cancel{(a)^m}\,a^{-1}}{\cancel{a^m}\,\cancel{b^m}\,b^{-1}} \quad \ldots\left\{\begin{array}{l}\because a^{m-1} = a^m\cdot a^{-1}\\ \text{and} b^{m-1} = b^m\, b^{-1}\end{array}\right.$$

$$\therefore \left(\frac{dy}{dx}\right)_{(a,b)} = -\frac{a^{-1}}{b^{-1}} = -\frac{b}{a} \quad \because a^{-1} = \frac{1}{a} \text{ and } \frac{1}{b^{-1}} = b$$

$$\therefore \text{Slope of the tangent at (a, b)} = \left(\frac{dy}{dx}\right)_{(a,b)} = \frac{-b}{a}$$

Step II : Substitute this value in Equation (1)

$\therefore$ Equation of the tangent passing through the point (a, b) is

$$y - b = \left(\frac{-b}{a}\right)(x-a)$$

$$\therefore a(y-b) = -b(x-a)$$

$$\therefore ay - ab = -bx + ba \quad \text{...by simplification}$$

$$\therefore bx + ay = \underbrace{ab + ba}_{2ab}$$

$$\therefore \mathbf{bx + ay = 2ab}$$

This is required equation of tangent.

Divide throughout by ab we get

$$\frac{bx+ay}{ab} = \frac{2ab}{ab}$$

$$\therefore \frac{\cancel{b}x}{a\cancel{b}} + \frac{\cancel{a}y}{\cancel{a}b} = \frac{2\cancel{ab}}{\cancel{ab}} \quad \because \frac{x+y}{a} = \frac{x}{a}+\frac{y}{a}$$

$$\therefore \frac{x}{a}+\frac{y}{b} = 2 \checkmark$$

....Hence proved.

Ex. 11.1.4 S-2014, 2 Marks

Find the inclination of the tangent to the curve $y = e^{2x}$ at (1, – 3)

☑ **Soln. :** Given equation of the curve is

$$y = e^{2x}$$

Differentiate w.r.t. x, we get

$$\frac{dy}{dx} = \underbrace{\frac{d}{dx} e^{2x}}_{e^{2x}\frac{d}{dx}(2x)} \qquad \text{...By derivative rule}$$

$$\therefore \frac{dy}{dx} = e^{2x} \underbrace{\frac{d}{dx}(2x)}_{2}$$

$$\therefore \frac{dy}{dx} = e^{2x}(2) = 2e^{2x}$$

$$\therefore \frac{dy}{dx} = 2e^{2x} \qquad \text{...(1)}$$

Now at point (1, – 3) i.e. at x = 1, y = – 3

$$\text{Equation (1)} \Rightarrow \left(\frac{dy}{dx}\right)_{(1,-3)} = \underbrace{2e^{2(1)}}_{2e^2} \qquad \text{... putting } x = 1$$

$$\therefore \left(\frac{dy}{dx}\right)_{(1,-3)} = 2e^2$$

We know,

$$\text{Slope of the tangent} = \frac{dy}{dx}$$

$$\therefore \text{Slope of the tangent at } (1,-3) = \left(\frac{dy}{dx}\right)_{(1,-3)}$$

$$= 2e^2 \qquad \text{...(2)}$$

Also, we know,

$$\text{Slope of tangent} = \tan\psi \qquad \text{...(3)}$$

Where ψ is the angle made by tangent with X-axis.

Equating RHS of Equations (2) and (3) we get

$$2e^2 = \tan\psi$$

$$\text{i.e. } \tan\psi = 2e^2$$

$$\therefore \psi = \tan^{-1}\underbrace{(2e^2)}_{14.7781} \qquad \left[\because \tan x = a \Rightarrow x = \tan^{-1} a\right]$$

$$\therefore \psi = \underbrace{\tan^{-1}(14.7781)}_{86.1288°} \qquad \text{...by table of values}$$

$$\therefore \psi = 86.1288° \approx 86.13°$$

∴ Inclination of tangent at point (1, – 3) is $\psi \approx 86.13°$ ✓ ...Ans.

Ex. 11.1.5 **S-2014, 2 Marks**

Find the point on the curve $y = 2x^2 - 6x$ where the tangent is parallel to the x-axis.

☑ **Soln. :**

Step I : Given equation of the curve is

$$y = 2x^2 - 6x \qquad \text{...(1)}$$

Differentiate both sides w.r.t. x we get,

$$\frac{dy}{dx} = \underbrace{\frac{d}{dx}[2x^2 - 6x]}_{}$$

$$= \frac{d}{dx}(2x^2) - \frac{d}{dx}(6x)$$

➤ **Using linearity property of derivative**

$$\text{...}\left[\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\right]$$

$$\therefore \frac{dy}{dx} = \underbrace{\frac{d}{dx}(2x^2)}_{2\frac{d}{dx}x^2} - \underbrace{\frac{d}{dx}(6x)}_{6\frac{d}{dx}x}$$

➤ **Using property of derivative**

$$\text{...}\left[\frac{d}{dx}\, k\, f(x) = k \frac{d}{dx} f(x)\right]$$

$$\therefore \frac{dy}{dx} = 2\underbrace{\frac{d}{dx}x^2}_{2x} - 6\underbrace{\frac{d}{dx}x}_{1}$$

$$\therefore \frac{dy}{dx} = \underbrace{2(2x)}_{4x} - \underbrace{6(1)}_{6} \qquad \text{...By derivative rule}$$

$$\therefore \frac{dy}{dx} = 4x - 6$$

$$\therefore \text{Slope of tangent} = 4x - 6 \qquad \text{...(2)}$$

Step II :

Also given, tangent is parallel to X-axis.

We know, slope of the X-axis = 0

Slope of two parallel lines are equal.

$$\therefore \text{Slope of tangent} = 0 \qquad \text{...(3)}$$

Step III : Equating RHS of Equations (2) and (3) we get

$$4x - 6 = 0$$

$$\therefore 4x = 6$$

$$\Rightarrow x = \frac{6}{4} = \frac{3}{2} \qquad \text{...Simplification}$$

$$\therefore \mathbf{x = \frac{3}{2}}$$

Now to find corresponding y, substitute x = 3/2 in Equation (1) we get,

$$y = 2\underbrace{\left(\frac{3}{2}\right)^2}_{9/4} - 6\left(\frac{3}{2}\right)$$

$$\therefore y = \cancel{2}\left(\frac{9}{\cancel{4}_2}\right) - \overset{3}{\cancel{6}}\left(\frac{3}{\cancel{2}}\right)$$

$$\therefore y = \frac{9}{2} - \underbrace{3(3)}_{9}$$

$$\therefore y = \frac{9}{2} - 9 = \frac{9 - 9 \times 2}{2} = \frac{9 - 18}{2}$$

$$\therefore y = -9/2 \qquad \text{...On simplification}$$

$$\therefore x = \frac{3}{2} \quad \text{and } y = \frac{-9}{2}$$

∴ Required point is, $(x, y) = \left(\frac{3}{2}, \frac{-9}{2}\right)$ ✓ ...Ans.

Ex. 11.1.6 (W-2016, 4 Marks)

Find equation of tangent to the circle $x^2 + y^2 + 6x - 6y - 7 = 0$ at a point it cuts the x-axis.

☑ **Soln. :**

Step : Given equation of circle is,

$$x^2 + y^2 + 6x - 6y - 7 = 0 \quad \text{…(1)}$$

Differentiate both sides w. r. to x, it gives,

$$\frac{d}{dx}[x^2 + y^2 + 6x - 6y - 7] = \frac{d}{dx}(0)$$

➤ **Using linearity property of derivative**

$$\text{…}\left[\frac{d}{dx}[a\,f(x) + b\,y(x)] = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)\right]$$

$$\therefore \underbrace{\frac{d}{dx}(x^2)}_{2x} + \underbrace{\frac{d}{dx}(y^2)}_{2y\frac{dy}{dx}} + 6\underbrace{\frac{d}{dx}(x)}_{1} - 6\underbrace{\frac{d}{dx}(y)}_{\frac{dy}{dx}} - \underbrace{\frac{d}{dx}(y)}_{0} = 0$$

$$2x + 2y\frac{dy}{dx} + 6 - 6\frac{dy}{dx} = 0$$

Collecting terms containing $\frac{dy}{dx}$ on L.H.S. and remaining on L.H.S.

$$2y\frac{dy}{dx} - 6\frac{dy}{dx} = -2x - 6$$

$$(2y - 6)\frac{dy}{dx} = -2(x + 3)$$

$$\frac{dy}{dx} = \frac{-2(x+3)}{2y - 6} = \frac{-\cancel{2}(x+3)}{\cancel{2}(y-3)}$$

$$\frac{dy}{dx} = -\frac{x+3}{y-3}$$

We know,

$$\text{Slope of the tangent} = \frac{dy}{dx}$$

$$\therefore \text{Slope of tangent} = \frac{dy}{dx} = -\frac{x+3}{y-3}$$

Step II : The curve cut to X-axis means y = 0

$\therefore \quad y = 0$ [Equation of X-axis]

Substitute y = 0 in Equation (1), it gives,

$$x^2 + 0 + 6x - 6(0) - 7 = 0$$

$$x^2 + 6x - 7 = 0$$

$$x^2 + 7x - x - 7 = 0 \quad \text{…}\left[\begin{array}{l}(7)(-1) = -7 \text{ and} \\ (7) + (-1) = 6\end{array}\right]$$

$$(x + 7)(x - 1) = 0$$

$$x + 7 = 0 \text{ and } x - 1 = 0$$

$$x = -7 \text{ and } x = 1$$

$\therefore$ Points of intersection of the curve with X-axis are (– 7, 0) and (1, 0)

$$\text{Since slope of tangent} = \frac{dy}{dx} = -\frac{x+3}{y-3} \quad \text{…(2)}$$

At point (– 7, 0)

$$\therefore \text{Slope of tangent} = \left(\frac{dy}{dx}\right)_{(-7,0)}$$

$$m = -\left(\frac{-7+3}{0-3}\right)$$

$$m = -\frac{4}{3}$$

$$\text{Slope of tangent} = m = \frac{-4}{3}$$

At point (1, 0)

$$\text{Slope of tangent} = \left(\frac{dy}{dx}\right)_{(1,0)}$$

$$m = -\left(\frac{1+3}{0-3}\right)$$

$$m = \frac{4}{3}$$

$$\text{Slope of tangent} = m = \frac{4}{3}$$

Step III : We know,

Equation of the tangent passing through (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

where m is slope of tangent at point (x_1, y_1)

$\therefore$ Equation of tangent passing through (– 7, 0) with slope $m = \frac{-4}{3}$ is,

$$y - 0 = \frac{-4}{3}(x - (-7))$$

$$3(y) = -4(x + 7)$$

$$3y = -4x - 28$$

$$\mathbf{4x + 3y = -28}$$

Equation of tangent passing though (1, 0) with slope $m = \frac{4}{3}$ is,

$$y - 0 = \frac{4}{3}(x - 1)$$

$$3(y) = 4(x - 1)$$

$$3y = 4x - 4$$

$\therefore$ $\mathbf{4x - 3y = 4}$ ✓ **…Ans.**

These are the equations of tangents to the circle where it cuts to X-axis.

Ex. 11.1.7 S-2013, 2 Marks

Find the equation of the curve whose slope is (x – 3) and which passes through (2, 0).

☑ **Soln. :**

Step I : We know

$$\text{Slope of tangent} = \frac{dy}{dx} \quad \text{…(1)}$$

Given slope of tangent = x – 3…(2)

$\therefore$ By equating Equations (1) and (2) we get

$$\frac{dy}{dx} = x - 3$$

i.e. $\quad dy = (x - 3)\,dx$ This is variable separable form

Step II :

∴ Integrate above Equation on both sides, we get

$$\int dy = \int (x-3)\,dx = \int x\,dx - \int 3\,dx$$

➢ **Using linearity property of integration**

$$\ldots\left[\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx\right]$$

$$\therefore \int dy = \int x\,dx - \int 3\,dx$$

$$y \quad \frac{x^{1+1}}{1+1} \quad -3x$$

➢ **Using property of integration**

$$\ldots\left[\begin{array}{l}\int x^n\,dx = \frac{x^{n+1}}{n+1},\ n = 1 \\ \text{and } \int k\,dx = k\int dx = k\,x\end{array}\right]$$

$$\therefore \quad y = \frac{x^{1+1}}{1+1} - 3x + c$$

↓ k

$$\therefore \quad \mathbf{y = \frac{x^2}{2} - 3x + c} \qquad \ldots(3)$$

Since this curve passing through the point (2, 0) is at x = 2 and y = 0

$$\text{Equation(3)} \Rightarrow 0 = \frac{(2)^2}{2} - 3(2) + c$$

$$0 = \frac{4}{2} - 3(2) + c$$

$$\therefore \quad 0 = 2 - 6 + c \ldots \text{on simplification}$$

$$-4$$

$$\therefore \quad 0 = -4 + c \Rightarrow c = 4$$

Put c = 4 in Equation (3) we get

$$y = \frac{x^2}{2} - 3x + 4$$

Multiple by 2 we get

$$2y = x^2 - 2(3x) + 2(4)$$

$$\therefore \quad \mathbf{2y = x^2 - 6x + 8} \checkmark \qquad \textbf{...Ans.}$$

This is required equation of curve

Ex. 11.1.8 **S-2012, 2 Marks**

Find the equation of curve whose slope is (2x + 5) and passes through the point (0, 2).

☑ **Soln. :** **Step I :** We know

$$\text{Slope of tangent} = \frac{dy}{dx} \qquad \ldots(1)$$

Given : Slope of tangent = 2x + 5 …(2)

∴ Equating Equations (1) and (2) we get

➢ **Using linearity property of integration**

$$\ldots\left[\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx\right]$$

$$\frac{dy}{dx} = (2x+5)$$

$$\therefore \quad dy = (2x+5)\,dx \quad \ldots\text{variable separable form}$$

Step II : Integrate both sides we get

$$\int dy = \int (2x+5)\,dx = \int 2x\,dx + \int 5\,dx$$

$$\therefore \int dy = \int 2x\,dx + \int 5\,dx$$

$$\therefore \int dy = 2\int x\,dx + 5\int dx \ldots\left[(\because \int k\,f(x)\,dx = k\int f(x)\,dx)\right]$$

$$y \quad \frac{x^2}{2} \quad x$$

$$\therefore \quad y = \cancel{2}\,\frac{x^3}{\cancel{2}} + 5x + c \ldots \text{by rule of Integration}$$

$$\therefore \quad \mathbf{y = x^3 + 5x + c} \qquad \textbf{...(3)}$$

Since, this curve passing through the point(0, 2) i.e. at x = 0 and y = 2 in Equation (3)

We get

$$2 = (0)^2 + 5(0) + c \Rightarrow 2 = 0 + 0 + c$$

$$\Rightarrow \mathbf{c = 2}$$

Put this c = 2 in Equation (3) we get

Equation (3) ⇒ $\mathbf{y = x^2 + 5x + 2}$ ✓ **...Ans.**

This is required equation of curve.

EXERCISE 11.2

Ex. 11.2.1 **(S-2016, 4 Marks)**

Find the maximum and minimum value of $y = x^3 - 9x^2 + 24x$.

☑ **Soln. :**

Step I :

$$\text{Given,} \quad y = x^3 - 9x^2 + 24x \qquad \ldots(1)$$

Differentiate w.r. to x, it gives,

$$\frac{dy}{dx} = 3x^2 - 9(2x) + 24(1) \text{ (By standard derivatives)}$$

$$\frac{dy}{dx} = 3x^2 - 18x + 24 \qquad \ldots(2)$$

Step II : We know,

To find maxima and minima of the curves y = f(x),

$$\text{Put} \quad \frac{dy}{dx} = 0 \quad \text{[By condition of maxima and minima]}$$

$$\therefore \quad 3x^2 - 18x + 24 = 0$$

$$3(x^2 - 6x + 8) = 0 \text{ (Common out 3)}$$

$3(x-4)(x-2) = 0$ (Note the factors)

$x - 4 = 0$ and $x - 2 = 0$

$x = 4$ $x = 2$

∴ Maxima or minima of y = f(x) are at point x = 4, x = 2

Step III : Again differentiate Equation (2) w.r. to x, it gives,

$$\frac{d^2y}{dx^2} = 3(2x) - 18(1) + 0$$

$$\frac{d^2y}{dx^2} = 6x - 18 \quad \text{...(3)}$$

At point x = 4 :

$$\left(\frac{d^2y}{dx^2}\right)_{x=4} = 6(4) - 18 \text{[Substitute x = 4 in Equation (3)]}$$

$$= 6 > 0$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{x=4} > 0 \quad \text{(Refer condition)}$$

∴ At point x = 4, y = f(x) has minima

$$\therefore y_{min} = f(4)$$

$$= (4)^3 - 9(4)^2 + 24(4)$$

[Substitute x = 4 in Equation (1)]

$$= 64 - 9(16) + 24(4)$$

$$y_{min} = 16$$

∴ Minima of y = f(x) = 16 and point of minima = (4, 16)

At point x = 2

$$\left(\frac{d^2y}{dx^2}\right)_{x=2} = 6(2) - 18$$

[Substitute x = 2 in Equation (3)]

$$= -6 < 0$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=2} < 0 \quad \text{(Refer condition)}$$

∴ At point x = 2, y = f(x) has maxima,

$$y_{max} = y(2)$$

$$= (2)^3 - 9(2)^2 + 24(2)$$

(Substitute x = 2 in Equation (1))

$$= 8 - 9(4) + 24(2)$$

$$y_{max} = 20$$

∴ Maxima of y = f(x) = 20 and point of maxima = (2, 20)

Ex. 11.2.2 S-2014, 4 Marks

Divide 100 into two parts such that their product is maximum.

☑ **Soln. :**

Step I : Divide 100 into two parts, as x and (100 – x)

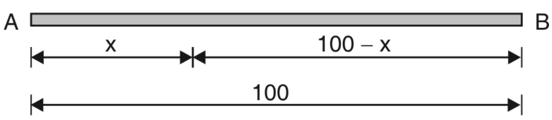

Fig. P. 11.2.2

Consider, Product = P = x (100 – x)

$$\therefore P = 100x - x^2 \quad \text{...(1)}$$

...On simplification

Differentiate w.r.t. x on both sides to Equation (1)

➤ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\right]$$

$$\text{Equation (1)} \Rightarrow \frac{d}{dx}P = \underbrace{\frac{d}{dx}(100x - x^2)}_{\frac{d}{dx}100x - \frac{d}{dx}x^2}$$

$$\therefore \frac{dP}{dx} = \underbrace{\frac{d}{dx}100x}_{100\frac{d}{dx}x} - \frac{d}{dx}x^2$$

➤ **Using property of derivative** $\ldots\left[\frac{d}{dx}k f(x) = k\frac{d}{dx}f(x)\right]$

$$\therefore \frac{dP}{dx} = 100\underbrace{\frac{d}{dx}x}_{1} - \underbrace{\frac{d}{dx}x^2}_{2x}$$

$$\therefore \frac{dP}{dx} = 100(1) - 2x \text{ by rule of derivative}$$

$$\therefore \frac{dP}{dx} = \underbrace{100}_{2\times 50} - 2x \quad \text{...(2)}$$

$$\therefore \frac{dP}{dx} = (2 \times 50) - 2x$$

$$\therefore \frac{dP}{dx} = 2[50 - x] \quad \text{...(3)}$$

Step II : We know,

To find maxima / minima of the product P

put $\frac{dP}{dx} = 0$ ...by condition of maxima / minima

$\therefore 100 - 2x = 0$ ... by Equation (2)

$\therefore (2 \times 50) - 2x = 0$

$\therefore 2[50 - x] = 0$ ...By taking 2 common

$\therefore 50 - x = 0$ ∵ mx = 0 ⇒ x = 0

$\therefore \underbrace{-x = -50}$ ...shifting 50 to RHS

$\therefore x = 50$

$$\therefore \frac{dP}{dx} = (2 \times 50) - 2x$$

$$\therefore \frac{dP}{dx} = 2[50 - x]$$

∴ Maxima / Minima of product P is at x = 50

Step III : Again differentiate Equation (2) w.r.t. x

We get

$$\frac{d}{dx}\left(\frac{dP}{dx}\right) = \frac{d}{dx}(100 - 2x)$$

$$\underbrace{\frac{d^2P}{dx^2}} \qquad \underbrace{\frac{d}{dx}100 - \frac{d}{dx}2x}$$

$$\frac{d^2P}{dx^2} = \underbrace{\frac{d}{dx}100}_{0} - \underbrace{\frac{d}{dx}2x}_{2}$$

$$\therefore \frac{d^2P}{dx^2} = 0 - 2 = -2 \quad \text{... by rule of derivative}$$

$$\frac{d^2P}{dx^2} = -2 < 0$$

$\therefore$ At point x = 50, P has maxima

$\because$ If $\frac{d^2y}{dx^2} < 0$ at $x = a$ then $f(x) = y$ has maxima at $x = 0$

Here a = 50

$\therefore$ x = 50 and 100 – x

= 100 – 50 = 50... putting x = 50

$\therefore$ Divide 100 as 50 and 50 so that their product is maximum.✓ ...**Ans.**

Ex. 11.2.3 (W-2016, S-2019, 4 Marks)

A metal wire of 40 cm long is bent to form a rectangle. Find its dimensions when its area is maximum.

☑ **Soln. :**

Step 1 : A metal wire 40 cm long is bent to form a rectangle.

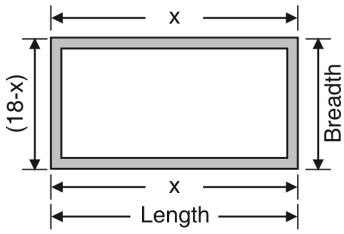

Fig. P. 11.2.3

$\therefore$ Two sides as length and two sides as breadth 40 cm.

Consider length of rectangle x cm

$\therefore$ Two sides length = x + x = 2x cm

$\therefore$ Two sides Breadth = (40 – 2x) cm

= 2(20 – x)

$\therefore$ Length of rectangle = x cm and

Breadth of a rectangle = (20 – x) cm

We know,

Area of rectangle = Length × Breadth

$\therefore$ A = x(20 – x)

A = $(20x - x^2)$

Differentiate w.r. to x, it gives

$$\frac{dA}{dx} = 20 - 2x \quad \text{...(2)}$$

Step II : we know

To find maxima of Area of rectangle

Put $\frac{dA}{dx} = 0$ [by condition of maxima/minimal]

$\therefore$ 20 – 2x = 0

– 2x = 20

$x = \frac{-20}{-2}$

$\therefore$ x = 10

$\therefore$ Area is maximum / minimum when x = 10

Step III : Again differentiate Equation (2) w.r.t to x. It gives

$$\frac{d^2A}{dx^2} = 0 - 2\,(1)$$

$$\frac{d^2A}{dx^2} = -2 \quad \text{(This is independent of x)}$$

At point x = 10 :

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ (Refer Condition)}$$

$\therefore$ By condition of Maxima / minima

$\therefore$ At point x = 10 Area (A) is maxima

$\therefore$ Length of rectangle = x = 10 cm

Breadth of rectangle = 20 – x = 20 – 10 cm = 10

$\therefore$ Area of rectangle is maximum

with length = 10 cm and Breadth = 10 cm. ✓ ...**Ans.**

Ex. 11.2.4 W-2013, Q. 3(b), W-18, 4 Marks

A manufacture can sell x items at price is of ₹ (330 – x) each. The cost of producing x items in ₹ $x^2 + 10x + 12$. How many items must be sold so that his profit is maximum.

☑ **Soln. :**

Step I : We know,

Profit = selling price – cost price

Given sell price of item = Rs. (330 – x)

Since, these are x items

$\therefore$ Total selling price = $\underbrace{x(330 - x)}_{330x - x^2}$

Also, Total cost of producing x items in Rs. is

= $x^2 + 10x + 12$

$\therefore$ Profit = $P = (330x - x^2) - (x^2 + 10x + 12)$

= $330x - x^2 - x^2 - 10x - 12$ ...on simplification

$\therefore$ P = $(-x^2 - x^2) + (330x - 10x) - 12$ ...Grouping terms

$\therefore$ P = $-2x^2 + 320x - 12$ ...(1)

Step II : We have find minima of profit (P)

By condition of maxima/minima

Put $\frac{dP}{dx} = 0$ ...(2)

Differentiate Equation (1) w.r.to x we get

➤ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\right]$$

$$\frac{d}{dx}P = \underbrace{\frac{d}{dx}[-2x^2 + 320x - 12]}_{\frac{d}{dx}(-2x^2) + \frac{d}{dx}(320x) - \frac{d}{dx}12}$$

$$\therefore \frac{d}{dx}P = \underbrace{\frac{d}{dx}(-2x^2)}_{-2(2x)} + \underbrace{\frac{d}{dx}(320x)}_{320} - \underbrace{\frac{d}{dx}12}_{0}$$

$$\therefore \frac{dP}{dx} = -2(2x) + 320 - 0 \quad \text{...by rule of derivative}$$

$$\therefore \frac{dP}{dx} = -4x + 320 \quad \text{...(3)}$$

$\therefore$ Put this value of $\frac{dP}{dx}$ in Equation (2)

Equation (2) becomes

$\Rightarrow -4x + 320 = 0$

$\Rightarrow -4x = -320$ ...shifting 320 to RHS

$\Rightarrow 4x = 320$

$\Rightarrow x = \frac{320}{4} = 80$

$\therefore$ Profit is maximum/minimum at point x= 80

Step III : Again differentiate Equation (2) w.r.t. x we get

$$\frac{d}{dx}\left(\frac{dP}{dx}\right) = \frac{d}{dx}(-4x + 320)$$

➤ **Using linearity property of derivative**

$$\ldots\left[\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)\right]$$

$$\frac{d^2P}{dx^2} = \underbrace{\frac{d}{dx}(-4x)}_{-4} + \underbrace{\frac{d}{dx}(320)}_{0}$$

$$\therefore \frac{d^2P}{dx^2} = -4 + 0 \quad \text{...by rule of derivative}$$

$$\therefore \frac{d^2P}{dx^2} = -4$$

Also at point x = 80

$$\left(\frac{d^2P}{dx^2}\right)_{x=80} = -4 < 0$$

$\therefore$ P = f(x) has maxima.

$\because$ by condition of maxima/minima

if $\frac{d^2y}{dx^2} < 0$ at x = a then y = f(x)

has maxima at x = a

Here a = 80,

At x = 80, P = f(x) has maxima.

$\therefore$ **For maximum profit, manufacture must be sold x = 80 items✓** **...Ans.**

Ex. 11.2.5 **W-2014, 4 Marks**

The perimeter of a rectangle is 100 m find the length of its sides when area of rectangle is maximum.

☑ **Soln. :**

Step I : Given perimeter of a rectangle = 100 m

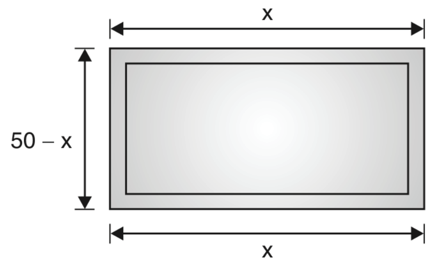

Fig. P. 11.2.5

Consider, Length of rectangle = x meter

Two sides length = x + x = 2x meter

Two sides breadth of rectangle = 2(50 – x)

= (100 – 2x) m ...(1)

Breadth of rectangle = (50 – x) m

We know,

Area of rectangle = Length × Breadth

$\therefore$ A = x ×(50 – x) ...By Equation (1)

$\therefore$ A = $50x - x^2$ ...(2)

Differentiate Equation (2) w.r.t. x we get

$$\frac{d}{dx}(A) = \underbrace{\frac{d}{dx}(50x - x^2)}_{\frac{d}{dx}(50x) - \frac{d}{dx}x^2} \quad \text{...by linearity property}$$

➤ **Using property of derivative**

$$\ldots\left[\frac{d}{dx}k\,f(x) = k\frac{d}{dx}f(x)\right]$$

$$\therefore \frac{dA}{dx} = 50\underbrace{\frac{d}{dx}x}_{1} - \underbrace{\frac{d}{dx}x^2}_{2x}$$

$$\therefore \frac{dA}{dx} = 50(1) - 2x \quad \text{....By rule of derivation}$$

$$\therefore \frac{dA}{dx} = 50 - 2x \quad \text{...(3)}$$

Step II :

We know,

To find maxima of Area of rectangle

Put $\frac{dA}{dx} = 0$ ...by condition of maxima/minima

$\therefore$ $50 - 2x = 0$ ...by Equation (3)

$\therefore$ $-2x = -50$ ...shifting 50 on RHS

$\therefore \quad 2x = 50$

$\therefore \quad x = \frac{50}{2} = 25$

$\therefore \quad x = 25$

$\therefore$ Area is maxima / minima when x = 25

Step III : Again Differentiate Equation (3) w.r.t. x

We get $\frac{d}{dx}\left(\frac{dA}{dx}\right) = \frac{d}{dx}(50 - 2x)$ …By Linearity property

$$\frac{d^2A}{dx^2} = \underbrace{\frac{d}{dx}50}_{0} - \underbrace{\frac{d}{dx}2x}_{2} \quad \text{…By rule of derivative}$$

$$\therefore \frac{d^2A}{dx^2} = \underbrace{0 - 2}_{-2}$$

$$\therefore \frac{d^2A}{dx^2} = -2$$

At point x = 25, $\frac{d^2A}{dx^2} = -2 < 0$

$\therefore$ By condition of maxima / minima A is maximum

at x = 25 ($\because$ y is maximum when $\frac{d^2A}{dx^2} < 0$ at x = a)

$\therefore$ At point x = 25 Equation (1) becomes,

Length of rectangle = x = 25 m and

Breadth of rectangle = (50 – x)

= 50 – 25 = 25 m

$\therefore$ **Area of rectangle is maximum**

With length = 25 meter

and Breadth = 25 meter ✓ **….Ans.**

EXERCISE 11.3

Ex. 11.3.1 **W-2015, 2 Marks**

Find the radius of curvature of $y = e^x$ at (0, 1).

OR

(Q. 2(d), S-19, 4 Marks)

Find the radius of curvature of the curve $y = e^x$ at the point where it crosses the Y-axis.

☑ **Soln. :**

Step I : We know,

The radius of curvature is,

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \quad \text{…(1)}$$

Step II : Given equation of curve is,

$$y = e^x \quad \text{…(2)}$$

Differentiate w.r.to x, it gives,

$$\frac{dy}{dx} = \frac{d}{dx}(e^x)$$

$$\frac{dy}{dx} = e^x \quad \text{…(3)}$$

Again differentiate w.r.to x,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(e^x)$$

$$\therefore \frac{d^2y}{dx^2} = e^x \quad \text{…(4)}$$

Step III : Now at given point, (0, 1)

$$\left(\frac{dy}{dx}\right)_{(0,1)} = e^0 = 1$$

(In Equation (3) substitute x = 0, y = 1) …(5)

And $\left(\frac{d^2y}{dx^2}\right)_{(0,1)} = e^0 = 1$

(In Equation (4) substitute x = 0, y = 1) …(6)

Step IV : Substitute values from Equations (5) and (6) in formula (1), it gives,

$$\text{Radius of curvature} = \rho = \frac{[1 + (1)^2]^{3/2}}{(1)}$$

$$= \frac{(1+1)^{3/2}}{1} = (2)^{3/2}$$

$$= 1.5874 \text{ units}$$

Radius of curvature = ρ = 1.5874 units ✓ **…Ans.**

Ex. 11.3.2 **(W-2014, 2 Marks**

Find the radius of curvature for the curve $y^2 = 4ax$ at point (a, 2a).

☑ **Soln. :** Step I : We know, the radius of curvature is,

$$\sigma = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \quad \text{…(1)}$$

First find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from the given Equation of the curve.

Step II : Given Equation of curve is

$$y^2 = 4ax \quad \text{…(2)}$$

Differentiate both sides w.r. to x we get,

$$\underbrace{\frac{d}{dx}(y^2)}_{2y\frac{dy}{dx}} = \underbrace{\frac{d}{dx}(4ax)}_{4a\frac{d}{dx}(x)}$$

➔ **Using property of derivative**

$$\text{…}\left[\frac{d}{dx}k\,f(x) = k\frac{d}{dx}f(x)\right]$$

$$\therefore 2y\frac{dy}{dx} = 4a\underbrace{\frac{d}{dx}(x)}_{(1)}$$

$$\therefore 2y\frac{dy}{dx} = 4a(1) \quad \text{…by rule of derivative}$$

$$\therefore \frac{dy}{dx} = \frac{^2\cancel{4}a}{\cancel{2}y} = \frac{2a}{y} \quad \text{...on simplification}$$

$$\therefore \frac{dy}{dx} = \frac{2a}{y} \quad \text{...(3)}$$

Again differentiate (3) w.r.t. to x we get

$$\underbrace{\frac{d}{dx}\left(\frac{dy}{dx}\right)}_{\frac{d^2y}{dx^2}} = \underbrace{\frac{d}{dx}\left(\frac{2a}{y}\right)}_{2a\frac{d}{dx}\left(\frac{1}{y}\right)}$$

➔ **Using property of derivative**

$$\text{...}\left[\frac{d}{dx} k\, f(x) = k \frac{d}{dx} f(x)\right]$$

$$\therefore \frac{d^2y}{dx^2} = 2a \underbrace{\frac{d}{dx}\left(\frac{1}{y}\right)}_{-\frac{1}{y^2}\frac{dy}{dx}}$$

…by derivative of composite function

$$\therefore \frac{d^2y}{dx^2} = 2a\left(-\frac{1}{y^2}\underbrace{\frac{dy}{dx}}_{\frac{2a}{y}}\right) \quad \text{...by Equation (3)}$$

$$\therefore \frac{d^2y}{dx^2} = 2a\underbrace{\left(-\frac{1}{y^2}\left(\frac{2a}{y}\right)\right)}_{\frac{-4a^2}{y^3}}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-4a^2}{y^3} \quad \text{...(4)}$$

Step III : At point (a, 2a)

Equation (3) becomes,

$$\left(\frac{dy}{dx}\right)_{(a,\,2a)} = \underbrace{\frac{2a}{2a}}_{1} \quad \text{...Put } y = 2a$$

$$\therefore \left(\frac{dy}{dx}\right)_{(a,\,2a)} = 1 \quad \text{...(5)}$$

and Equation (4) becomes,

$$\frac{d^2y}{dx^2} = \frac{-4a^2}{\underbrace{(2a)^3}_{8a^3}} \quad \text{...Put } y = 2a$$

$$\frac{d^2y}{dx^2} = \frac{-\cancel{4}\cancel{a^2}}{_2\cancel{8}a^{\cancel{3}}} = \frac{-1}{2a}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-1}{2a} \quad \text{...(6)}$$

Step IV : Substitute values of Equations (5) and (6) in Equation (1) we get,

Radius of curvature

$$\rho = \frac{\left[1 + (1)^2\right]^{3/2}}{\frac{-1}{2a}}$$

$$\therefore \rho = \frac{[1 + 1]^{3/2}}{-1/2a}$$

$$\therefore \rho = \left(\frac{-2a}{1}\right)(2)^{3/2} \quad \text{...on simplification}$$

$$\therefore \rho = -(2)(2)^{3/2}\, a \quad \text{...on simplification}$$

$$\therefore \rho = -\left(2^{1+3/2}\right) a$$

$$\therefore \rho = -2^{\frac{2+3}{2}} a = \underbrace{-2^{5/2}}_{-5.6568} a$$

$$\therefore \rho = -5.6568 \text{ a units} \quad \text{...on simplification}$$

∴ Radius of curvature

ρ = **– 5.6568 a units** ✓ **…Ans.**

Chapter Ends…

❑❑❑

Chapter 12 : STATISTICS

Exercise 12.1

Ex. 12.1.1 (W-14, 2 Marks)

Find the range of the following distribution :

2, 3, 1, 10, 6, 31, 17, 20, 24

☑ **Soln. :** We know, Range = L – S …(1)

Where L = Largest value; S = Smallest value

Given, 2, 3, 1, 10, 6, 31, 17, 20, 24

Observe that, from these values,

Largest value = L = 31

Smallest value = S = 1

∴ Range = 31 – 1 [From Equation (1)]

Range = 30

Ex. 12.1.2 (S-13, 2 Marks)

Find the range of the following distribution :

3, 6, 10,1, 15, 16, 21, 19, 18.

☑ **Soln. :** We know, Range = L – S …(1)

Where, L = Largest value of distribution

S = Smallest value of distribution

Given distribution, 3, 6, 10,1, 15, 16, 21, 19, 18

Observe that, from these values,

∴ Largest value = L = 21

Smallest value = S = 1

∴ Range = 21 – 1 [From Equation (1)]

Range = 20

Ex. 12.1.3 (S-16, 2 Marks.

Find the range of the following data :

800, 725, 750, 900, 925, 910, 1000, 790, 870, 920

☑ **Soln. :** Given,

the range 800, 725, 750, 900, 925, 910, 1000, 790, 870, 920

Range = Largest value – Smallest value = 1000 – 725

=275

Ex. 12.1.4 (S-14, 2 Marks.

Find the range and coefficient of range for the following data : 120, 100, 130, 50, 150

☑ **Soln. :** We know, Range = L – S …(1)

and coefficient of range $= \frac{L-S}{L+S}$ …(2)

Where, L = Largest value of data

S = Smallest value of data

Given, 120, 100, 130, 50, 150

Observe that from these values.

Largest value = L = 150
Smallest value = S = 50 } …(3)

Substitute these values in Equations (1) and (3)

∴ Range = L – S =150 – 50

Range = 100

$$\text{Coefficient of range} = \frac{L-S}{L+S}$$

$$= \frac{150-50}{150+50} \text{ [Values from Equation (3)]}$$

$$= \frac{100}{200} = \frac{1}{2}$$

Coefficient of Range = 0.5

Ex. 12.1.5 (Q. 1(f), W-17, 2 Marks)

Find range and coefficient of range for the data : 120, 50, 90, 100, 180, 200, 150, 40, 80

☑ **Soln. :**

We know Range = L – S

Where, L = Largest value

S = Smallest value

Given, 120, 50, 90, 100, 180, 200, 150, 40, 80

Observe that from these values

Largest value = L = 200

Smallest value = S= 40

Range = L – S = 200 – 40

Range = 160 **…Ans.**

Ex. 12.1.6 (S-2013, 4 Marks)

Find the range and coefficient of range of the following data

Age (in years)	10-19	20-29	30-39	40-49	50-59	60-69	70-79
Frequency	03	61	223	137	53	19	04

☑ **Soln. :** We know, Range=L – S …(1)

and coefficient of range $= \frac{L-S}{L+S}$ …(2)

Where, L = Largest value of data

S = Smallest value of data

Given,

Age (x_i) (in years)	9.5-19.5	19.5-29.5	29.5-39.5	39.5-49.5	49.5-59.5	59.5-69.5	69.5-79.5
Frequency (f_i)	03	61	223	137	53	19	04

Observe that from this tabular values,

Largest value of x_i = L = 79.5
Smallest value of x_i = S = 9.5 …(3)

(Note that do not consider values of f_i for range and coefficient of range.)

(i) Range

From Equation (1) and Equation (3),

Range = L – S = 79.5 – 9.5

Range = 70

(ii) Coefficient of range

From Equation (2) and Equation (3),

$$\text{Coefficient of Range} = \frac{L-S}{L+S} = \frac{79.5-9.5}{79.5+9.5} = \frac{70}{89} = 0.787$$

∴ **Coefficient of Range = 0.787**

Ex. 12.1.7 **(W-14, S-17, 4 Marks)**

Find the range and coefficient of range for following data.

Marks	20-29	30-39	40-49	50-59	60-69	70-79	80-89	90-99
No. of students	10	15	16	20	21	22	09	08

☑ **Soln. :** We know, Range = L – S …(1)

and Coefficient of Range $= \frac{L-S}{L+S}$ …(2)

Where, L = Largest value of data

S = Smallest value of data

Given,

Marks (x_i)	19.5-29.5	29.5-39.5	39.5-49.5	49.5-59.5	59.5-69.5	69.5-79.5	79.5-89.5	89.5-99.5
No. of students (f_i)	10	15	16	20	21	22	09	08

Observe that from this tabular values,

Largest value of x_i = L = 99.5
Smallest value of x_i = S = 19.5 …(3)

(i) Range : From Equation (1) and Equation (3),

Range = L – S = 99.5 – 19.5 = 80

Range = 80

(ii) Coefficient of range

From Equation (2) and Equation (3),

$$\text{Coefficient of Range} = \frac{L-S}{L+S} = \frac{99.5-19.5}{99.5+19.5} = \frac{80}{119} = 0.672$$

∴ **Coefficient of Range = 0.672**

Exercise 12.2

Ex. 12.2.1 : Calculate mean deviation about mean and median for the Data: 1, 2, 3, 4, 5, 6, 7, 8, 9,

☑ **Soln. :**

Mean deviation about mean :

We know,

$$\text{Mean} = \bar{x} = \frac{\sum x_i}{N} \text{ and}$$

$$\text{Mean deviation about mean} = \frac{\sum |d_i|}{N}$$

Now by given data obtain following table as,

x_i	$d_i = x_i - \bar{x}$	$\lvert d_i \rvert$
1	– 4	4
2	– 3	3
3	– 2	2
4	– 1	1
5	0	0
6	1	1
7	2	2
8	3	3
9	4	4
$\sum x_i = 45$		$\sum d_i = 20$

(a) $\bar{x} = \frac{\sum x_i}{N} = \frac{45}{9} = 5$ (Use this value in table for calculation)

(b) Mean deviation about mean $= \frac{\sum |d_i|}{n} = \frac{20}{9} = 2.22$

Mean deviation about median

We know,

$$\text{Median} = M = \frac{\sum x_i}{N} \text{ and}$$

$$\text{Median deviation about mean} = \frac{\sum |d_i|}{N}$$

Arrange data 1, 2, 3, 4, [5], 6, 7, 8, 9,

N = odd = 9

$$\text{Median} = \left(\frac{N+1}{2}\right)^{th} \text{ place observation}$$

$$\text{Median} = \left(\frac{9+1}{2}\right)^{th} = 5^{th} \text{ place}$$

Median = 5

Now by given data obtain following table as,

x_i	$d_i = x_i - M$	$\lvert d_i \rvert$
1	– 4	4
2	– 3	3
3	– 2	2
4	– 1	1
5	0	0
6	1	1
7	2	2
8	3	3
9	4	4
$\sum x_i = 45$		$\sum \lvert d_i \rvert = 20$

$$\text{Mean deviation about median} = \frac{\sum \lvert d_i \rvert}{N} = \frac{20}{9} = 2.22$$

Ex. 12.2.2 : Calculate mean deviation from mean and median.

x_i	10	11	12	13	14
F_i	3	12	18	12	3

☑ **Soln. :**

(a) Mean deviation abut mean

Mean deviation about mean:

We know, $\quad \text{Mean} = \bar{x} = \dfrac{\sum f_i x_i}{f_i}$ and

$$\text{Mean deviation about mean} = \frac{\sum f_i \lvert d_i \rvert}{\sum f_i}$$

Now by given data obtain following table as,

a_i	f_i	$f_i x_i$	$x_i - \bar{x} = x_i - 12$	$\lvert d_i \rvert = \lvert x_i - \bar{x} \rvert$	$f_i \lvert d_i \rvert$
10	3	30	– 2	2	6
11	12	132	– 1	1	12
12	18	216	0	0	00
13	12	156	1	1	12
14	3	42	2	2	6
	$\sum f_i = 48$	$\sum f_i x_i = 576$			$\sum f_i \lvert d_i \rvert = 36$

(i) $\text{mean} = \bar{x} = \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{576}{48} = 12 \left(\begin{array}{c} \text{Use this value} \\ \text{in table for calculation} \end{array} \right)$

(ii) $\text{Mean deviation abut mean} = \dfrac{\sum f_i \lvert d_i \rvert}{\sum f_i} = \dfrac{36}{48} = 0.75$

(b) Mean deviation about median:

We know, $\quad \text{Median} = M = \dfrac{\sum x_i}{N}$ and

$$\text{Median deviation about mean} = \frac{\sum f_i \lvert d_i \rvert}{\sum f_i}$$

Now by given data obtain following table as,

x_i	f_i	C.F.	$x_i - M$	$\lvert d_i \rvert = \lvert x_i - M \rvert$	$f_i \lvert d_i \rvert$
10	3	3	– 2	2	6
11	12	15	– 1	1	12
12	18	33	0	0	00
13	12	45	1	1	12
14	3	48	2	2	6
	$\sum f_i = 48$				$\sum f_i \lvert d_i \rvert = 36$

Here N = 48 even

If number of observation is even, N = even

$$\text{Median} = \frac{\left(\frac{N}{2}\right)^{th} + \left(\frac{N}{2} + 1\right)^{th}}{2}$$

$$\text{Median} = \frac{\left(\frac{48}{2}\right)^{th} + \left(\frac{48}{2} + 1\right)^{th}}{2} = \frac{24 + 25}{2} = 24.5$$

It lies between C.F. (class frequency) 15 and 33

$\therefore$ For C.F. = 33 corresponding value of $x_i = 12$

Median = M = 12

Now,

$$\text{Mean deviation (M.D.)} = \frac{\sum f_i \lvert d_i \rvert}{\sum f_i} \qquad \text{[From Equation (1)]}$$

$$= \frac{36}{48} = 0.75 \qquad \text{[Values from table]}$$

Mean deviation (M.D.) from median = 0.75

Ex.12.2.3 **(Q. 2(d) S-18, 4 Marks)**

Calculate the mean deviation about the mean of the following data :

3, 6, 5, 7, 10,12, 15, 18.

☑ **Soln. :**

We have,

$$\text{Mean deviation about mean} = \frac{\sum \lvert x - \bar{x} \rvert}{N} = \frac{\sum \lvert d_i \rvert}{N} \qquad \text{…(1)}$$

$$\text{Where } \bar{x} = \text{mean and } \bar{x} = \frac{\sum x_i}{N} \qquad \text{…(2)}$$

Now from given data obtain the table as

x_i	$d_i = x - \bar{x}$, $\bar{x} = 9.5$	$\lvert d_i \rvert = \lvert x_i - \bar{x} \rvert$
3	– 6.5	6.5
5	– 4.5	4.5
6	– 3.5	3.5
7	– 2.5	2.5
10	0.5	0.5
12	2.5	2.5
15	5.5	5.5
18	8.5	8.5
$\sum x_i = 76$		$\sum \lvert d_i \rvert = 34$

Here, N = 8

Mean $\bar{x} = \dfrac{\sum x_i}{N}$ [From equation (2)]

$\bar{x} = \dfrac{76}{8} = 9.5$

(Use this value in table for calculations)

∴ Mean deviation about mean $= \dfrac{\sum \lvert d_i \rvert}{N}$ [From equation (1)]

$= \dfrac{34}{8} = 4.25$

∴ Mean deviation (M.D.) about mean = 4.25 **…Ans.**

Ex. 12.2.4 W-11, W-12, W-13, S-14, 4 Marks

Q. 6(a)(i), S-19, 3 Marks

Find the mean deviation from mean of the following distribution.

Marks	0-10	10-20	20-30	30-40	40-50
No. of student	05	08	15	16	06

☑ **Soln. :** We know,

Mean deviation about mean $= \dfrac{\sum f_i \lvert x_i - \bar{x} \rvert}{\sum f_i} = \dfrac{\sum f_i \lvert d_i \rvert}{N}$ …(1)

Where, $\bar{x}$ = Mean and $\bar{x} = \dfrac{\sum f_i x_i}{\sum f_i}$ …(2)

Now from given data obtain the table as :

Marks (class interval)	Middle value x_i	No. of students f_i	$f_i x_i$	$d_i = x_i - \bar{x}$, $\bar{x} = 27$	$\lvert d_i \rvert = \lvert x_i - \bar{x} \rvert$	$f_i \lvert d_i \rvert$
0-10	5	05	25	– 22	22	110
10-20	15	08	120	– 12	12	96
20-30	25	15	375	– 2	2	30
30-40	35	16	560	8	8	128
40-50	45	06	270	18	18	108
		$\sum f_i = N = 50$	$\sum f_i x_i = 1350$			$\sum f_i \lvert d_i \rvert = 472$

Mean $= \dfrac{\sum f_i x_i}{\sum f_i}$ [From Equation (2)]

$= \dfrac{1350}{50} = \mathbf{27}$ (Use this value in table for calculation)

∴ Mean deviation about mean

$= \dfrac{\sum f_i \lvert d_i \rvert}{N}$ [From Equation (1)]

$= \dfrac{472}{50}$ (Values from table)

$= 9.44$

∴ **Mean Deviation about mean = 9.44**

Ex. 12.2.5 (S-15, 4 Marks)

Calculate mean deviation about mean of the following distribution :

x_i	3	4	5	6	7	8
f_i	4	9	10	8	6	3

☑ **Soln. :** We know, Mean deviation about mean

$$= \dfrac{\sum f_i \lvert x_i - \bar{x} \rvert}{\sum f_i} = \dfrac{\sum f_i \lvert d_i \rvert}{N} \quad …(1)$$

where, $\bar{x}$ = mean and $\bar{x} = \dfrac{\sum f_i x_i}{\sum f_i}$ …(2)

Now from given data obtain the table as :

x_i	f_i	$f_i x_i$	$d_i = x_i - \bar{x}$, $\bar{x} = 5.3$	$\lvert d_i \rvert = \lvert x_i - \bar{x} \rvert$	$f_i \lvert d_i \rvert$
3	4	12	– 2.3	2.3	9.2
4	9	36	– 1.3	1.3	11.2
5	10	50	– 0.3	0.3	3

x_i	f_i	$f_i x_i$	$d_i = x_i - \bar{x}$, $\bar{x} = 5.3$	$\lvert d_i \rvert = \lvert x_i - \bar{x} \rvert$	$f_i \lvert d_i \rvert$
6	8	48	0.7	0.7	5.6
7	6	42	1.7	1.7	10.2
8	3	24	2.7	2.7	8.1
	$\sum f_i = N = 40$	$\sum f_i x_i = 212$			$\sum f_i \lvert d_i \rvert = 47.8$

$\therefore$ Mean = $\bar{x} = \dfrac{\sum f_i x_i}{\sum f_i}$ [From Equation (2)]

$\bar{x} = \dfrac{212}{40} = \mathbf{5.3}$ (Use this value in table for calculation)

$\therefore$ Mean deviation about mean

$$= \frac{\sum f_i \lvert d_i \rvert}{\sum f_i} = \frac{47.8}{40} = 1.195 \approx 1.20$$

$\therefore$ **Mean Deviation (M.D.) about mean = 1.20**

☞ Exercise 12.3

Ex. 12.3.1 (S-15, 2 Marks)

If mean is 82.5, standard deviation is 7.2, find co-efficient of variance.

☑ **Soln. :** We know,

Co-efficient of variation is,

$$\text{C.V.} = \frac{\text{S.D.}}{\text{A.M.}} \times 100 = \frac{\sigma}{\bar{x}} \times 100 \qquad \ldots(1)$$

Given, Mean $= \bar{x} = 82.5$ and S.D. $= \sigma = 7.2$

Substitute these values in Equation (1)

$\therefore$ Co-efficient of variation $= \dfrac{7.2}{82.5} \times 100 = 8.73$

C.V. = 8.73

Ex. 12.3.2 (W-11, 2 Marks)

Find the standard deviation for the following data :

1, 2, 3, 4, 5, 6, 7, 8, 9

☑ **Soln. :** We know,

Standard Deviation (S.D.) $\sigma = \sqrt{\dfrac{\sum d_i^2}{N}}$ …(1)

Where, $d_i = x_i - \bar{x}$, $\bar{x}$ = Mean and $\bar{x} = \dfrac{\sum x_i}{N}$ …(2)

For the given data obtain table as :

x_i	$d_i = x_i - \bar{x}$, $\bar{x} = 5$	d_i^2
1	– 4	16
2	– 3	9
3	– 2	4
4	– 1	1
5	0	0
6	1	1
7	2	4
8	3	9
9	4	16
$\sum x_i = 45$		$\sum d_i^2 = 60$

$\therefore$ Mean = $\bar{x} = \dfrac{\sum x_i}{N} = \dfrac{45}{9}$ [Value $\sum x_i$ from table and N = total numbers = 9]

$\therefore \bar{x} = 5$ Use this value in table for calculation

Now, Standard Deviation (S.D.)

$$\sigma = \sqrt{\frac{60}{9}} \quad \text{(From Equation (1) and values from table)}$$

$$= \sqrt{\frac{20}{3}}$$

$\sigma = 2.58$

$\therefore$ **S.D. = σ = 2.58**

Ex. 12.3.3 (W-08, 4 Marks)

Find the standard deviation of the following frequency table.

Weekly expenditure below Rs.	05	10	15	25
No. of students	06	16	20	46

☑ **Soln. :**

We know,

Standard Deviation (S.D.)

$$\sigma = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}} = \sqrt{\frac{\sum f_i d_i^2}{N}} \qquad \ldots(1)$$

Where, $\bar{x}$ = Mean and $\bar{x} = \dfrac{\sum f_i x_i}{\sum f_i}$ …(2)

From the given data obtain the table as :

Weekly expenditure below Rs. (x_i)	No. of students f_i	$f_i x_i$	$d_i = (x_i - \bar{x})$, $\bar{x} = 18.64$	d_i^2	$f_i d_i^2$
05	06	30	– 13.64	186.05	1116.3
10	16	160	– 8.64	74.65	1194.4
15	20	300	– 3.64	13.25	265

Weekly expenditure below Rs. (x_i)	No. of students f_i	$f_i x_i$	$d_i = (x_i - \bar{x})$ $\bar{x} = 18.64$	d_i^2	$f_i d_i^2$
25	46	1150	6.36	40.45	1860.7
	$\sum f_i = N = 88$	$\sum f_i x_i = 1640$			$\sum f_i d_i^2 = 4436.4$

$$\text{Mean} = \bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{1640}{88} = \mathbf{18.64}$$

(Use this value in table for further calculation)

Standard deviation (S.D.)

$$\sigma = \sqrt{\frac{\sum fi\ di2}{N}} \quad \text{[From Equation (1)]}$$

$$= \sqrt{\frac{4436.4}{88}} \quad \text{(Values from table)}$$

$$= \sqrt{50.4136}$$

$$\sigma = 7.1$$

$\therefore$ **S.D. = 7.1**

Method II : Standard Deviation (S.D.)

$$\sigma = \sqrt{\frac{\sum f_i x_i^2}{N} - (\bar{x})^2}$$

Weekly expenditure below Rs. (x_i)	No. of students f_i	$f_i x_i$	x_i^2	$f_i x_i^2$
05	06	30	25	150
10	16	160	100	1600
15	20	300	225	4500
25	46	1150	625	28750
	$\sum f_i = N = 88$	$\sum f_i x_i = 1640$		$\sum f_i x_i^2 = 35000$

$$\text{Mean} = \bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{1640}{88}$$

$$\bar{x} = \mathbf{18.64}$$

(Use this value in table for further calculation)

Standard deviation (S.D.)

$$\sigma = \sqrt{\frac{\sum f_i x_i^2}{N} - (\bar{x})^2}$$

$$= \sqrt{\frac{35000}{88} - (18.64)^2} = \sqrt{50.28} = 7.09$$

$$\sigma \approx 7.1$$

S.D. = 7.1

Ex. 12.3.4 **(S-07, W-12, S-13, W-13, S-17,4 Marks)**

Calculate the standard deviation for following distribution.

Class interval	0-5	5-10	10-15	15-20	20-25	25-30	30-35	35-40
Frequency	3	5	9	15	20	16	10	2

Also find : (i) variance

(ii) Coefficient of variance

☑ **Soln. :** We know, Standard deviation (S.D.) is

$$\sigma = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}} = \sqrt{\frac{\sum f_i d_i^2}{N}} \quad \text{…(1)}$$

Where, $\bar{x}$ = Mean and $\bar{x} = \dfrac{\sum f_i x_i}{\sum f_i}$ …(2)

From the given data obtain the table as :

Class interval (C.I.)	Middle values (M.V) x_i	Frequency f_i	$f_i x_i$	$d_i = x_i - \bar{x}$ Where $\bar{x} = 21.38$	d_i^2	$f_i d_i^2$
0-5	2.5	3	7.5	− 18.88	356.45	1069.35
5-10	7.5	5	37.5	− 13.88	192.65	963.25
10-15	12.5	9	112.5	− 8.88	78.85	709.65
15-20	17.5	15	262.5	− 3.88	15.05	225.75
20-25	22.5	20	450	1.12	1.25	25
25-30	27.5	16	440	6.12	37.45	599.2
30-35	32.5	10	325	11.12	123.65	1236.5
35-40	37.5	2	75	16.12	259.85	519.7
		$\sum f_i = 80$	$\sum f_i x_i = 1710$			$\sum f_i d_i^2 = 5348.4$

$$\text{Mean} = \bar{x} = \frac{\sum f_i x_i}{\sum f_i} \quad \text{(From Equation (2))}$$

$$= \frac{1710}{80}$$

$$\bar{x} = \mathbf{21.38} \quad \text{(use this value in table)}$$

Now, Standard deviation (S.D.) is,

$$\sigma = \sqrt{\frac{\sum f_i d_i^2}{\sum f_i}} \quad \text{[From Equation (1)]}$$

$$= \sqrt{\frac{5348.4}{80}} = \sqrt{68.855}$$

$$\sigma = 8.176$$

i.e. S.D. = σ = 8.18

Method II : We know, Standard deviation (S.D.) is,

$$\left.\begin{aligned}\sigma &= \sqrt{\frac{\sum f_i x_i^2}{N} - (\bar{x})^2} \\ \text{Where} \quad \bar{x} &= \frac{\sum f_i x_i}{\sum f_i};\ N = \sum f_i\end{aligned}\right\} \quad ...(3)$$

Class interval (C.I.)	Middle values (M.V)	Frequency f_i	$f_i x_i$	x_i^2	$f_i x_i^2$
0-5	2.5	3	7.5	6.25	18.75
5-10	7.5	5	37.5	56.25	281.25
10-15	12.5	9	112.5	156.25	1406.25
15-20	17.5	15	262.5	306.25	4593.75
20-25	22.5	20	450	506.25	10125
25-30	27.5	16	440	756.25	12100
30-35	32.5	10	325	1056.25	10562.5
35-40	37.5	2	75	1406.25	2812.5
		$\sum f_i = 80$	$\sum f_i x_i = 1710$		$\sum f_i x_i^2 = 41900$

$$\text{Mean} = \bar{x} = \frac{1710}{80} \quad \text{(From Equation (2))}$$

$$\bar{x} = \mathbf{21.38} \quad \text{(use this value in table)}$$

Standard deviation (S.D.) is,

$$\sigma = \sqrt{\frac{41900}{80} - (21.38)^2}$$

$$\sigma = \sqrt{523.75 - 457.1044} = \sqrt{66.6456}$$

$$\sigma = 8.164$$

$$\therefore \ \textbf{S.D.} = \sigma = \mathbf{8.164}$$

Ex. 12.3.5 (S-08,W-15, 4 Marks)

Find the standard deviation of the following.

Class interval	0-20	20-40	40-60	60-80	80-100
Frequency	20	130	220	70	60

☑ **Soln. :** We know,

Standard deviation (S.D.) is,

$$\sigma = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}} = \sqrt{\frac{\sum f_i d_i^2}{N}} \quad ...(1)$$

$$\text{Where, } \bar{x} = \text{Mean and } \bar{x} = \frac{\sum f_i x_i}{\sum f_i} \quad ...(2)$$

$$N = \sum f_i$$

From the given data obtain the table as :

Class interval (C.I.)	Middle values (M.V) x_i	$d_i = x_i - \bar{x}$ where $\bar{x} = 50.8$	Frequency f_i	$f_i x_i$	d_i^2	$f_i d_i^2$
0-20	10	– 40.8	20	200	1664.64	33292.8
20-40	30	– 20.8	130	3900	432.64	56243.2
40-60	50	– 0.8	220	11000	0.64	140.8
60-80	70	19.2	70	4900	368.64	25804.8
80-100	90	39.2	60	5400	1536.64	92198.4
		$\sum f_i x_i = 25400$	$\sum f_i = 500$			$\sum f_i d_i^2 = 207680$

$$\text{Mean} = \bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{25400}{500}$$

$$\text{Mean} = \bar{x} = \mathbf{50.8} \quad \text{(Use this value in table)}$$

Now, Standard deviation (S.D.) is,

$$\sigma = \sqrt{\frac{\sum f_i d_i^2}{\sum f_i}} = \sqrt{\frac{207680}{500}}$$

$$\sigma = \sqrt{415.36}$$

$$\sigma = 20.38$$

$$\textbf{i.e. S.D.} = \sigma = \mathbf{20.38}$$

Method II : We know, Standard deviation (S.D.) is,

$$\sigma = \sqrt{\frac{\sum f_i x_i^2}{\sum f_i} - (\bar{x})^2}$$

$$\text{Where, } \bar{x} = \text{Mean and } \bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

Class interval (C.I.)	Middle values (M.V) x_i	Frequency f_i	$f_i x_i$	x_i^2	$f_i x_i^2$
0-20	10	20	200	100	2000
20-40	30	130	3900	900	117000
40-60	50	220	11000	2500	550000
60-80	70	70	4900	4900	343000
80-100	90	60	5400	8100	486000
		$\sum f_i = 500$	$\sum f_i x_i = 25400$		$\sum f_i x_i^2 = 1498000$

$$\text{Mean} = \bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{25400}{500}$$

$$\text{Mean} = \bar{x} = \mathbf{50.8} \quad \text{(Use this value in table)}$$

$$\sigma = \sqrt{\frac{1498000}{500} - (50.8)^2}$$

$$= \sqrt{2996 - 2580.64} = \sqrt{415.36}$$

$$\sigma = 20.38$$

$$\therefore \quad \textbf{S.D.} = \sigma = \mathbf{20.38}$$

Ex. 12.3.6 **(W-2014,4 Marks.**

Find the variance and coefficient of variance for the following distribution.

Class interval	20-25	25-30	30-35	35-40	40-45	45-50	50-55	55-60	60-65
Frequency	35	30	50	90	75	60	35	25	15

☑ **Soln. :** We know, Standard deviation (S.D.) is,

$$\sigma = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}} = \sqrt{\frac{\sum f_i d_i^2}{N}} \quad \ldots(1)$$

$$\text{Where,} \quad \bar{x} = \text{Mean} \quad \text{and} \ \bar{x} = \frac{\sum f_i x_i}{\sum f_i} \quad \ldots(2)$$

$$N = \sum f_i$$

From the given data obtain the table as :

Class interval C.I.	Middle value x_i	Frequency f_i	$f_i x_i$	$d_i = x_i - \bar{x}$ where $\bar{x} = 25$	d_i^2	$f_i d_i^2$
20-25	22.5	25	562.5	– 18.58	345.2164	8630.41
25-30	27.5	30	825	– 13.58	184.4164	5532.492
30-35	32.5	50	1625	– 8.58	73.6164	3680.82
35-40	37.5	90	3375	– 3.58	12.8164	1153.476
40-45	42.5	75	3187.5	1.42	2.0164	151.23
45-50	47.5	60	2850	6.42	41.2164	2472.984
50-55	52.5	35	1837.5	11.42	130.4164	4564.574
55-60	57.5	25	1437.5	16.42	269.6164	6740.41
60-65	65.5	15	937.5	21.42	458.8164	6882.246
		$\sum f_i = 405$	$\sum f_i x_i = 16637.5$			$\sum f_i d_i^2 = 39808.64$

$$\text{Mean} = \bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{16637.5}{405} \quad \text{(Value from table)}$$

$$\therefore \quad \text{Mean} \quad \bar{x} = \mathbf{41.08} \quad \text{(Use this value in table)}$$

Now, Standard deviation (S.D.) is,

$$\sigma = \sqrt{\frac{\sum f_i d_i^2}{\sum f_i}} = \sqrt{\frac{39808.64}{405}}$$

$$\sigma = \sqrt{98.29} = 9.91$$

$$\textbf{S.D.}\ \sigma = \mathbf{9.91}$$

Method II

We know, Standard deviation (S.D.) is,

$$\sigma = \sqrt{\frac{\sum f_i x_i^2}{\sum f_i} - (\bar{x})^2}$$

$$\text{Where,} \quad \bar{x} = \text{Mean} \quad \text{and} \quad \bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

Class interval C.I.	Middle value x_i	Frequency f_i	$f_i x_i$	x_i^2	$f_i x_i^2$
20-25	22.5	25	562.5	506.25	12656.25
25-30	27.5	30	825	756.25	22687.5
30-35	32.5	50	1625	1056.25	52812.5
35-40	37.5	90	3375	1406.25	126562.5
40-45	42.5	75	3187.5	1806.25	135468.5
45-50	47.5	60	2850	2256.25	135375
50-55	52.5	35	1837.5	2756.25	96468.75
55-60	57.5	25	1437.5	3306.25	82656.25
60-65	65.5	15	937.5	3906.25	58593.75
		$\sum f_i = 405$	$\sum f_i x_i = 16637.5$		$\sum f_i x_i^2 = 723281.3$

$$\sigma = \sqrt{\frac{723281.3}{405} - (41.08)^2} = \sqrt{98.31} \quad \ldots(3)$$

$$\therefore \textbf{S.D.} = \sigma = \mathbf{9.91}$$

We know,

$$\text{Variance} = (\text{S.D.})^2 = \sigma^2$$

$$\text{Variance} = 98.31$$

$$\text{Also,} \quad \text{Coefficient of variance} = \frac{\text{S.D.}}{\text{Mean}} \times 100 = \frac{\sigma}{\bar{x}} \times 100$$

$$= \frac{9.91}{41.08} \times 100 = 24.12$$

$$\therefore \quad \textbf{Coefficient of variance} = \mathbf{24.12}$$

Ex. 12.3.7 **(W-2011, 4 Marks)**

Following are the marks obtained by two students X and Y :

Marks obtained by X	44	80	76	48	52	72	68	56	60	64
Marks obtained by Y	48	75	54	60	63	69	72	51	57	56

Which of the students is more consistent ?

☑ **Soln. :** To find consistency first find coefficient of variance.

We know,

$$\text{Coefficient of variance} = \frac{\text{S.D.}}{\text{Mean}} \times 100 = \frac{\sigma}{\bar{x}} \times 100$$

$$\text{Standard deviation (S.D.) is, } \sigma = \sqrt{\frac{(x_i - \bar{x})^2}{N}}$$

Where, $\bar{x}$ = Mean and $\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$

For X

x_i	$x_i - \bar{x}$ Where, $\bar{x} = 62$	$(x_i - \bar{x})^2$
44	– 18	324
80	18	324
76	14	196
48	– 14	196
52	– 10	100
72	10	100
68	6	36
56	– 6	36
60	– 2	4
64	2	4
$\sum = 620$		$\sum = 1320$

$$\text{Mean} = \bar{x} = \frac{\sum x_i}{N} = \frac{620}{10}$$

Mean $= \bar{x} = 62$

Standard deviation (S.D.) is,

$$\sigma = \sqrt{\frac{1320}{10}} = 11.49$$

$\therefore$ **S.D. = σ = 11.49**

$\therefore$ Coefficient of variation (C.V.)

$$= \frac{\text{S.D.}}{\text{Mean}} \times 100 = \frac{\sigma}{\bar{x}} \times 100 = \frac{11.49}{62} \times 100 = 18.53$$

$\therefore$ **Coefficient of variance = 18.53**

For Y

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
48	– 12.5	156.25
75	14.5	210.25
54	– 6.5	42.25
60	– 0.5	0.25
63	2.5	6.25
69	8.5	72.25
72	11.5	132.25
51	– 9.5	90.25
57	– 3.5	12.25
56	– 4.5	20.25
$\sum = 605$		$\sum = 742.5$

$$\text{Mean} = \bar{x} = \frac{\sum x_i}{N} = \frac{605}{10}$$

Mean $= \bar{x} = 60.5$

Now, Standard deviation (S.D.),

$$\sigma = \sqrt{\frac{742.5}{10}} = 8.62$$

S.D. = σ = 8.62

$$\therefore \text{Coefficient of variation} = \frac{\text{S.D.}}{\text{Mean}} \times 100 = \frac{\sigma}{\bar{x}} \times 100$$

$$= \frac{8.62}{60.5} \times 100 = 14.25$$

Coefficient of variation of X>Coefficient of variation of Y

$\therefore$ Consistency of X < Consistency of Y

$\therefore$ **Y student is more consistent.**

Ex. 12.3.8 **(S-15, 4 Marks)**

Two sets of observations are given below :

Set - I	Set - II
$\bar{x} = 82.5$	$\bar{x} = 98.75$
$\sigma = 7.3$	$\sigma = 8.35$

Which of two sets is more consistent.

☑ **Soln. :** We know, Coefficient of variation $= \frac{\text{S.D.}}{\text{Mean}} \times 100$

For set I : Coefficient of variation $= \frac{7.3}{82.5} \times 100 = 8.85$

For set II : Coefficient of variation $= \frac{8.35}{98.75} \times 100 = 8.46$

$\therefore$ Coefficient of variation of Set I > coefficient of variation of set II

$\therefore$ Consistency of Set I < Consistency of Set II

$\therefore$ **Set II is more consistent**

Chapter Ends…

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